

# Second Order Multiscale Stochastic Volatility Asymptotics: Stochastic Terminal Layer Analysis & Calibration

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## Abstract

Multiscale stochastic volatility models have been developed as an efficient way to capture the principle effects on derivative pricing and portfolio optimization of randomly varying volatility. The recent book Fouque, Papanicolaou, Sircar and Sølna (2011, CUP) analyzes models in which the volatility of the underlying is driven by two diffusions – one fast mean-reverting and one slow-varying, and provides a *first order* approximation for European option prices and for the implied volatility surface, which is calibrated to market data. Here, we present the full *second order* asymptotics, which are considerably more complicated due to a terminal layer near the option expiration time. We find that, to second order, the implied volatility approximation depends quadratically on log-moneyness, capturing the convexity of the implied volatility curve seen in data. We introduce a new probabilistic approach to the terminal layer analysis needed for the derivation of the second order singular perturbation term, and calibrate to S&P 500 options data.

## 1 Introduction

Stochastic volatility models relax the constant volatility assumption of the Black-Scholes model for option pricing by allowing volatility to fluctuate randomly. As a result, they are able to capture some of the well-known features of the implied volatility surface, such as the volatility smile and skew. While some single-factor diffusion stochastic volatility models such as Heston's [14], enjoy wide success due to the existence of semi-analytic pricing formula for European options, it is known that such models are not adequate to match implied volatility levels across all strikes and maturities; see, for instance, [11]. Numerous empirical studies have identified at least a fast time scale in stock price volatility on the order of days, as well as a slow scale on the order of months, for example [2, 5, 15, 17]. This has motivated the development of multiscale stochastic volatility models, in which instantaneous volatility levels are controlled by multiple driving factors running on different time scales.

A class of multiscale stochastic volatility models is analyzed in [7], where an approximation for European options and their induced implied volatilities is derived, which can capture the overall level of implied volatility, its skew across strike prices and its term-structure over a wide range of maturities. However, the analysis there is limited to a first order approximation, which cannot pick up the slight convexity of the observed equity implied volatility surface. In this paper we extend the results of [7] to second order. This extension is non-trivial, as it requires a careful terminal layer analysis, which we approach probabilistically. For some related multiscale perturbation techniques in European option pricing, we refer for instance to [3]

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and [4] (spectral methods), [16] (matched asymptotic expansions), [1], [13] and [10] (Malliavian calculus), and [21] (inner-outer expansions).

Our second order results allow us to capture the slight convexity of the implied volatility skew. Additionally, we are able to maintain analytic tractability which is important for calibration to data, as we demonstrate. Of course, numerous asymptotic regimes have been analyzed in recent years for the option pricing problem in incomplete markets: see [8], [12] and [19] for some references. Here our focus is not just on deriving and proving convergence of the approximation in the appropriate limits, but in disentangling the calibration procedure that results from it. Compared to the first order theory, this is much more involved as there are many more group parameters and basis functions that have to be accommodated to implied volatility data. Despite the increase in complexity, we show this can be implemented successfully.

The rest of this paper proceeds as follows. In Section 2, we describe the class of multiscale stochastic volatility models that we will work with. Using a formal singular and regular perturbation analysis, we derive a pricing approximation which is valid for any European-style option. We establish the accuracy of our pricing approximation in Theorem 2.4. In Section 3, we present an explicit formula for the implied volatility surface induced by our option pricing approximation. Additionally, we show how a parameter reduction, crucial for calibration purpose, can be achieved with no loss of accuracy. In Section 3.2, we outline a procedure for calibrating the class of multiscale stochastic volatility models to the empirically observed implied volatility surface of liquid calls and puts. We carry out this calibration procedure on call and put data taken from the S&P500 index. Section 4 concludes.

## 2 Second Order Option Pricing Asymptotics

We consider the class of multiscale stochastic volatility models studied in [8]. Let  $X$  denote the price of a non-dividend-paying asset whose dynamics under the historical probability measure  $\mathbb{P}$  is defined by the following system of stochastic differential equations (SDEs):

$$\left. \begin{aligned} dX_t &= \mu X_t dt + f(Y_t, Z_t) X_t dW_t^x, \\ dY_t &= \frac{1}{\varepsilon} \alpha(Y_t) dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) dW_t^y, \\ dZ_t &= \delta c(Z_t) dt + \sqrt{\delta} g(Z_t) dW_t^z. \end{aligned} \right\}$$

Here,  $(W^x, W^y, W^z)$  are  $\mathbb{P}$ -Brownian motions with correlation structure

$$d\langle W^x, W^y \rangle_t = \rho_{xy} dt, \quad d\langle W^x, W^z \rangle_t = \rho_{xz} dt, \quad d\langle W^y, W^z \rangle_t = \rho_{yz} dt,$$

where  $(\rho_{xy}, \rho_{xz}, \rho_{yz})$  satisfy  $|\rho_{xy}|, |\rho_{xz}|, |\rho_{yz}| < 1$  and  $1 + 2\rho_{xy}\rho_{xz}\rho_{yz} - \rho_{xy}^2 - \rho_{xz}^2 - \rho_{yz}^2 \geq 0$ , which guarantees that the correlation matrix of the Brownian motions is positive-semidefinite. The asset  $X$  has geometric growth rate  $\mu$  and stochastic volatility  $f(Y_t, Z_t)$  which is driven by two factors,  $Y$  and  $Z$ . Under the physical measure, the infinitesimal generators of  $Y$  and  $Z$  are scaled by factors of  $1/\varepsilon$  and  $\delta$  respectively. Thus,  $\varepsilon > 0$  and  $1/\delta > 0$  represent the intrinsic time-scales of these processes. We will work in the regime where  $\varepsilon \ll 1$  and  $\delta \ll 1$  so that  $Y$  and  $Z$  represent fast- and slow-varying factors of volatility respectively. Most importantly, we assume the fast factor is *mean-reverting*, that is that  $Y$  is an ergodic process with a unique invariant distribution  $\Pi$  under  $\mathbb{P}$ , which is independent of  $\varepsilon$ .

Under the risk-neutral pricing measure  $\mathbb{P}$  (chosen by the market) the dynamics are described by

$$\left. \begin{aligned} dX_t &= r X_t dt + f(Y_t, Z_t) X_t d\widetilde{W}_t^x, \\ dY_t &= \left( \frac{1}{\varepsilon} \alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}} \Lambda(Y_t, Z_t) \beta(Y_t) \right) dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t) d\widetilde{W}_t^y, \\ dZ_t &= \left( \delta c(Z_t) - \sqrt{\delta} \Gamma(Y_t, Z_t) g(Z_t) \right) dt + \sqrt{\delta} g(Z_t) d\widetilde{W}_t^z, \end{aligned} \right\} \quad (2.1)$$

where  $(\widetilde{W}^x, \widetilde{W}^y, \widetilde{W}^z)$  are  $\widetilde{\mathbb{P}}$ -Brownian motions with the same correlation structure as between their  $\mathbb{P}$ -counterparts, and  $r$  is the risk-free rate of interest. The functions  $\Lambda(y, z)$  and  $\Gamma(y, z)$  represent market prices of volatility risk, which we have assumed such as to preserve the Markov structure of  $(X, Y, Z)$  and of  $(Y, Z)$  by itself.

Consider a European option with expiration date  $T$  and payoff  $h(X_T)$ . The no-arbitrage price of this option at time  $t < T$  can be expressed as a discounted expectation of the option payoff

$$P^{\varepsilon, \delta}(t, X_t, Y_t, Z_t) = \widetilde{\mathbb{E}} \left[ e^{-r(T-t)} h(X_T) \middle| X_t, Y_t, Z_t \right].$$

Here,  $\widetilde{\mathbb{E}}$  denotes an expectation taken under the pricing measure  $\widetilde{\mathbb{P}}$ . Note that we have used the Markov property of  $(X, Y, Z)$  to express the price of the option as a function  $P^{\varepsilon, \delta}(t, x, y, z)$  of the current time  $t$  and the state variables  $(X_t, Y_t, Z_t)$ . Using the Feynman-Kac formula, one finds that  $P^{\varepsilon, \delta}$  satisfies the following partial differential equation (PDE) and terminal condition:

$$\mathcal{L}^{\varepsilon, \delta} P^{\varepsilon, \delta} = 0, \quad P^{\varepsilon, \delta}(T, x, y, z) = h(x), \quad (2.2)$$

where, introducing the notation

$$\mathcal{D}_k = x^k \partial_{x \dots x}^k, \quad k = 1, 2, \dots, \quad (2.3)$$

the operator  $\mathcal{L}^{\varepsilon, \delta}$  is given by

$$\mathcal{L}^{\varepsilon, \delta} = \left( \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) + \sqrt{\delta} \left( \frac{1}{\sqrt{\varepsilon}} \mathcal{M}_3 + \mathcal{M}_1 \right) + \delta \mathcal{M}_2,$$

with

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2} \beta^2(y) \partial_{yy}^2 + \alpha(y) \partial_y, \\ \mathcal{L}_1 &= \rho_{xy} \beta(y) f(y, z) \mathcal{D}_1 \partial_y - \beta(y) \Lambda(y, z) \partial_y, \\ \mathcal{L}_2 &= \partial_t + \frac{1}{2} f^2(y, z) \mathcal{D}_2 + r \mathcal{D}_1 - r, \\ \mathcal{M}_3 &= \rho_{yz} \beta(y) g(z) \partial_{yz}^2, \\ \mathcal{M}_1 &= \rho_{xz} g(z) f(y, z) \mathcal{D}_1 \partial_z - g(z) \Gamma(y, z) \partial_z, \\ \mathcal{M}_2 &= \frac{1}{2} g^2(z) \partial_{zz}^2 + c(z) \partial_z. \end{aligned}$$

For general  $(f, \alpha, \beta, \Lambda, c, g, \Gamma)$  no explicit solution to (2.2) exists and we seek an asymptotic approximation for the option price. The fast factor asymptotics is a singular perturbation problem, while the slow factor expansion is a regular perturbation. Thus, the small- $\varepsilon$  and small- $\delta$  regime gives rise to a combined singular-regular perturbation about the  $\mathcal{O}(1)$  operator  $\mathcal{L}_2$ . We expand  $P^{\varepsilon, \delta}$  in powers of  $\sqrt{\varepsilon}$  and  $\sqrt{\delta}$  as follows

$$P^{\varepsilon, \delta}(t, x, y, z) = \sum_{j \geq 0} \sum_{i \geq 0} \sqrt{\varepsilon}^i \sqrt{\delta}^j P_{i,j}(t, x, y, z).$$

This is a formal series expansion, and our approach is to find  $P_{i,j}$  for  $i + j \leq 2$  and establish an accuracy result for the truncated series. We point out that we are working within an infinite-dimensional family of models since the functions  $(f, \Lambda, \Gamma)$  are unspecified: the 18 group parameters that are found in Section 2.7 and calibrated in Section 3.2 contain specific moments of these functions identified by the asymptotic analysis.

Because we are performing a dual expansion in half-integer powers of  $\varepsilon$  and  $\delta$ , we must decide which of these parameters we will expand in first. We choose to perform a regular perturbation expansion with respect to  $\delta$  first. Then, within each of the equations that result from the regular perturbation analysis, we will perform a singular perturbation expansion with respect to  $\varepsilon$ . As the combined regular-singular perturbation expansion is quite lengthy, to aid the reader, we provide a summary of the key results in equation (2.41), located at the end of this section.

## 2.1 Regular Perturbation Analysis

The regular perturbation expansion proceeds by expanding  $\mathcal{L}^{\varepsilon,\delta}$  and  $P^{\varepsilon,\delta}$  in powers of  $\sqrt{\delta}$  as follows

$$\mathcal{L}^{\varepsilon,\delta} = \mathcal{L}^\varepsilon + \sqrt{\delta} \mathcal{M}^\varepsilon + \delta \mathcal{M}_2, \quad P^\varepsilon = \sum_{j \geq 0} \sqrt{\delta}^j P_j^\varepsilon, \quad (2.4)$$

where from (2)

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2, \quad \mathcal{M}^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \mathcal{M}_3 + \mathcal{M}_1, \quad P_j^\varepsilon = \sum_{i \geq 0} \sqrt{\varepsilon}^i P_{i,j}^\varepsilon. \quad (2.5)$$

Inserting (2.4) into (2.2) and collecting terms of like-powers of  $\sqrt{\delta}$ , we find that the lowest order equations of the regular perturbation expansion are

$$\mathcal{O}(1) : \quad 0 = \mathcal{L}^\varepsilon P_0^\varepsilon, \quad (2.6)$$

$$\mathcal{O}(\sqrt{\delta}) : \quad 0 = \mathcal{L}^\varepsilon P_1^\varepsilon + \mathcal{M}^\varepsilon P_0^\varepsilon, \quad (2.7)$$

$$\mathcal{O}(\delta) : \quad 0 = \mathcal{L}^\varepsilon P_2^\varepsilon + \mathcal{M}^\varepsilon P_1^\varepsilon + \mathcal{M}_2 P_0^\varepsilon. \quad (2.8)$$

Within each of these three equations, we now perform a singular perturbation analysis with respect to  $\varepsilon$ .

## 2.2 Singular Perturbation Analysis of the $\mathcal{O}(1)$ Equation (2.6)

Beginning with the  $\mathcal{O}(1)$  equation, we insert expansions (2.5) into (2.6) and collect terms of like-powers of  $\sqrt{\varepsilon}$ . The resulting  $\mathcal{O}(1/\varepsilon)$  and  $\mathcal{O}(1/\sqrt{\varepsilon})$  equations are:

$$\mathcal{O}(1/\varepsilon) : \quad 0 = \mathcal{L}_0 P_{0,0},$$

$$\mathcal{O}(1/\sqrt{\varepsilon}) : \quad 0 = \mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_{0,0}.$$

We note that all terms in  $\mathcal{L}_0$  and  $\mathcal{L}_1$  take derivatives with respect to  $y$ . Thus, if we choose  $P_{0,0}$  and  $P_{1,0}$  to be independent of  $y$ , the above equations will automatically be satisfied. Hence, we seek solutions of the form

$$P_{0,0} = P_{0,0}(t, x, z), \quad P_{1,0} = P_{1,0}(t, x, z),$$

i.e., no  $y$ -dependence. Continuing the asymptotic analysis, the  $\mathcal{O}(1)$ ,  $\mathcal{O}(\sqrt{\varepsilon})$  and  $\mathcal{O}(\varepsilon)$  equations are:

$$\mathcal{O}(1) : \quad 0 = \mathcal{L}_0 P_{2,0} + \cancel{\mathcal{L}_1 P_{1,0}} + \mathcal{L}_2 P_{0,0}, \quad (2.9)$$

$$\mathcal{O}(\sqrt{\varepsilon}) : \quad 0 = \mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0}, \quad (2.10)$$

$$\mathcal{O}(\varepsilon) : \quad 0 = \mathcal{L}_0 P_{4,0} + \mathcal{L}_1 P_{3,0} + \mathcal{L}_2 P_{2,0}, \quad (2.11)$$

where we have used the fact that  $\mathcal{L}_1 P_{1,0} = 0$ .

Equations (2.9), (2.10) and (2.11) are Poisson equations of the form

$$0 = \mathcal{L}_0 P + \chi. \quad (2.12)$$

By the Fredholm alternative, equation (2.12) admits a solution  $P$  only if  $\chi$  is in the orthogonal complement of the null space of the adjoint operator  $\mathcal{L}_0^*$ . Since the unique invariant distribution  $\Pi$  satisfies  $\mathcal{L}_0^* \Pi = 0$ , this leads to the solvability or *centering condition*:

$$\langle \chi \rangle := \int \chi(y) \Pi(dy) = 0. \quad (2.13)$$

Applying the centering conditions to equations (2.9), (2.10) and (2.11), and using the fact that  $P_{0,0}$  and  $P_{1,0}$  do not depend on  $y$ , we find

$$\mathcal{O}(1) : \quad 0 = \langle \mathcal{L}_2 \rangle P_{0,0}, \quad (2.14)$$

$$\mathcal{O}(\sqrt{\varepsilon}) : \quad 0 = \langle \mathcal{L}_1 P_{2,0} \rangle + \langle \mathcal{L}_2 \rangle P_{1,0}, \quad (2.15)$$

$$\mathcal{O}(\varepsilon) : \quad 0 = \langle \mathcal{L}_1 P_{3,0} \rangle + \langle \mathcal{L}_2 P_{2,0} \rangle, \quad (2.16)$$

where the operator  $\langle \mathcal{L}_2 \rangle$ , given by

$$\langle \mathcal{L}_2 \rangle = \partial_t + \frac{1}{2} \bar{\sigma}^2(z) \mathcal{D}_2 + r \mathcal{D}_1 - r, \quad (2.17)$$

with

$$\bar{\sigma}^2(z) := \langle f^2(\cdot, z) \rangle = \int f^2(y, z) \Pi(dy).$$

We observe that  $\langle \mathcal{L}_2 \rangle$  is the Black-Scholes pricing operator with *effective averaged* volatility  $\bar{\sigma}(z) = \sqrt{\bar{\sigma}^2(z)}$  in which the level  $z$  of the slow factor appears as a parameter.

Expanding the terminal condition in (2.2) leads to the terminal conditions

$$\mathcal{O}(1) : \quad P_{0,0}(T, x, z) = h(x), \quad (2.18)$$

$$\mathcal{O}(\sqrt{\varepsilon}) : \quad P_{1,0}(T, x, z) = 0. \quad (2.19)$$

We denote the solution to (2.14) with terminal condition (2.18) by

$$P_{0,0}(t, x, z) = P_{BS}(t, x; \bar{\sigma}(z)),$$

the Black-Scholes price of the option with volatility  $\bar{\sigma}(z)$ , maturity  $T$ , and payoff function  $h$ .

In order to make use of equation (2.15), we need an expression for  $\langle \mathcal{L}_1 P_{2,0} \rangle$ . Using (2.14), we re-write (2.9) as follows

$$\mathcal{L}_0 P_{2,0} = -\mathcal{L}_2 P_{0,0} = -(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_{0,0} = -\frac{1}{2} (f^2 - \langle f^2 \rangle) \mathcal{D}_2 P_{0,0}.$$

Introducing a solution  $\phi(y, z)$  to the Poisson equation

$$\mathcal{L}_0 \phi = f^2 - \langle f^2 \rangle, \quad (2.20)$$

we deduce the following expression for  $P_{2,0}$ :

$$P_{2,0}(t, x, y, z) = -\frac{1}{2} \phi(y, z) \mathcal{D}_2 P_{0,0}(t, x, z) + F_{2,0}(t, x, z), \quad (2.21)$$

where  $F_{2,0}(t, x, z)$  is independent of  $y$ .

**Remark.** The form of (2.21) shows that the natural terminal condition  $P_{2,0}(T, x, y, z) = 0$  is not enforceable because the singular perturbation with respect to the fast factor creates a terminal layer near  $t = T$ . However, as we will demonstrate in Section 2.6, the ergodic theorem enables us to impose the *averaged* terminal condition

$$\mathcal{O}(\varepsilon) : \quad \langle P_{2,0}(T, x, \cdot, z) \rangle = 0. \quad (2.22)$$

and to obtain the desired accuracy of our pricing approximation. In fact, we will see that this is the *only* appropriate choice for proof of convergence.

**Remark.** The solution of the Poisson equation (2.20) is defined up to a constant in  $y$ . We choose this constant by imposing the condition

$$\langle \phi(\cdot, z) \rangle = 0, \quad (2.23)$$

and we will show in Section 2.6 that this choice does not affect the accuracy of our pricing approximation.

Inserting (2.21) into (2.15) yields the following PDE for  $P_{1,0}$

$$\langle \mathcal{L}_2 \rangle P_{1,0} = -\langle \mathcal{L}_1 P_{2,0} \rangle = -\left\langle \left( \rho_{xy} \beta f \mathcal{D}_1 \partial_y - \beta \Lambda \partial_y \right) \left( -\frac{1}{2} \phi \mathcal{D}_2 P_{0,0} + F_{2,0} \right) \right\rangle = -\mathcal{V} P_{0,0}, \quad (2.24)$$

where the  $z$ -dependent operator  $\mathcal{V}$  is given by

$$\mathcal{V}(z) = V_3(z) \mathcal{D}_1 \mathcal{D}_2 + V_2(z) \mathcal{D}_2,$$

and we introduce the notation

$$V_2(z) = \frac{1}{2} \langle \beta(\cdot) \Lambda(\cdot, z) \partial_y \phi(\cdot, z) \rangle, \quad V_3(z) = -\frac{1}{2} \rho_{xy} \langle \beta(\cdot) f(\cdot, z) \partial_y \phi(\cdot, z) \rangle.$$

The solution  $P_{1,0}$  of the PDE (2.24) with terminal condition (2.19) will be given in Proposition 2.3.

To determine  $P_{2,0}$ , given by (2.21), we need a PDE and terminal condition for the unknown function  $F_{2,0}$ . The terminal condition  $F_{2,0}(T, x, z) = 0$  is imposed by averaging (2.21), and using (2.22) and (2.23).

**Proposition 2.1.** *The function  $F_{2,0}(t, x, z)$  satisfies the following PDE and terminal condition*

$$\langle \mathcal{L}_2 \rangle F_{2,0} = -\mathcal{A} P_{0,0} - \mathcal{V} P_{1,0}, \quad F_{2,0}(T, x, z) = 0, \quad (2.25)$$

where the  $z$ -dependent operator  $\mathcal{A}$  is given by

$$\begin{aligned} \mathcal{A}(z) &= A_2(z) \mathcal{D}_1^2 \mathcal{D}_2 + A_1(z) \mathcal{D}_1 \mathcal{D}_2 + A_0(z) \mathcal{D}_2 + A(z) \mathcal{D}_2^2, \\ A_2(z) &= \frac{1}{2} \rho_{xy}^2 \langle \beta(\cdot) f(\cdot, z) \partial_y \psi_1(\cdot, z) \rangle, \\ A_1(z) &= -\frac{1}{2} \rho_{xy} (\langle \beta(\cdot) \Lambda(\cdot, z) \partial_y \psi_1(\cdot, z) \rangle + \langle \beta(\cdot) f(\cdot, z) \partial_y \psi_2(\cdot, z) \rangle), \\ A_0(z) &= \frac{1}{2} \langle \beta(\cdot) \Lambda(\cdot, z) \partial_y \psi_2(\cdot, z) \rangle, \\ A(z) &:= -\frac{1}{4} \left( \langle \phi(\cdot, z) f^2(\cdot, z) \rangle - \langle \phi(\cdot, z) \rangle \langle f^2(\cdot, z) \rangle \right), \end{aligned}$$

and  $\psi_1(y, z)$  and  $\psi_2(y, z)$  satisfy

$$\mathcal{L}_0 \psi_1 = \beta f \partial_y \phi - \langle \beta f \partial_y \phi \rangle, \quad \mathcal{L}_0 \psi_2 = \beta \Lambda \partial_y \phi - \langle \beta \Lambda \partial_y \phi \rangle, \quad (2.26)$$

*Proof.* The proof is in appendix A. □

The solution  $F_{2,0}$  of the PDE with terminal condition (2.25) will be given in Proposition 2.3. This is as far as we will take the asymptotic analysis of the  $\mathcal{O}(1)$  equation (2.6).

### 2.3 Singular Perturbation Analysis of the $\mathcal{O}(\sqrt{\delta})$ Equation (2.7)

Proceeding as in Section 2.2, we insert expansions (2.5) into (2.7) and collect term of like-powers of  $\sqrt{\varepsilon}$ . The resulting  $\mathcal{O}(\sqrt{\delta}/\varepsilon)$  and  $\mathcal{O}(\sqrt{\delta}/\sqrt{\varepsilon})$  equations are:

$$\begin{aligned} \mathcal{O}(\sqrt{\delta}/\varepsilon) : \quad & 0 = \mathcal{L}_0 P_{0,1}, \\ \mathcal{O}(\sqrt{\delta}/\sqrt{\varepsilon}) : \quad & 0 = \mathcal{L}_0 P_{1,1} + \mathcal{L}_1 P_{0,1} + \cancel{\mathcal{M}_3 P_{0,0}}, \end{aligned}$$

where we have used  $\mathcal{M}_3 P_{0,0} = 0$  since  $\mathcal{M}_3$  contains  $\partial_y$  and  $P_{0,0}$  is independent of  $y$ . Recalling that all terms in  $\mathcal{L}_0$  and  $\mathcal{L}_1$  also contain  $\partial_y$ , we seek solutions  $P_{0,1}$  and  $P_{1,1}$  of the form

$$P_{0,1} = P_{0,1}(t, x, z), \quad P_{1,1} = P_{1,1}(t, x, z).$$

Continuing the asymptotic analysis, the  $\mathcal{O}(\sqrt{\delta})$  and  $\mathcal{O}(\sqrt{\delta}\sqrt{\varepsilon})$  equations are:

$$\mathcal{O}(\sqrt{\delta}) : \quad 0 = \mathcal{L}_0 P_{2,1} + \mathcal{L}_1 P_{1,1} + \mathcal{L}_2 P_{0,1} + \mathcal{M}_3 P_{1,0} + \mathcal{M}_1 P_{0,0}, \quad (2.27)$$

$$\mathcal{O}(\sqrt{\delta}\sqrt{\varepsilon}) : \quad 0 = \mathcal{L}_0 P_{3,1} + \mathcal{L}_1 P_{2,1} + \mathcal{L}_2 P_{1,1} + \mathcal{M}_3 P_{2,0} + \mathcal{M}_1 P_{1,0}. \quad (2.28)$$

Equations (2.27) and (2.28) are Poisson equations of the form (2.12). Applying the centering condition (2.13) to (2.27) and (2.28) yields

$$\mathcal{O}(\sqrt{\delta}) : \quad 0 = \langle \mathcal{L}_2 \rangle P_{0,1} + \langle \mathcal{M}_1 \rangle P_{0,0}, \quad (2.29)$$

$$\mathcal{O}(\sqrt{\delta}\sqrt{\varepsilon}) : \quad 0 = \langle \mathcal{L}_1 P_{2,1} \rangle + \langle \mathcal{L}_2 \rangle P_{1,1} + \langle \mathcal{M}_3 P_{2,0} \rangle + \langle \mathcal{M}_1 \rangle P_{1,0}. \quad (2.30)$$

We also impose the following terminal conditions

$$\mathcal{O}(\sqrt{\delta}) : \quad P_{0,1}(T, x, z) = 0, \quad (2.31)$$

$$\mathcal{O}(\sqrt{\delta}\sqrt{\varepsilon}) : \quad P_{1,1}(T, x, z) = 0. \quad (2.32)$$

The PDE (2.29) and terminal condition (2.31) can be used to find an expression for  $P_{0,1}$ , which will be given in Proposition 2.3.

The operator  $\langle \mathcal{M}_1 \rangle$  appearing in (2.29) is given by

$$\langle \mathcal{M}_1 \rangle = \rho_{xz} g \langle f \rangle \mathcal{D}_1 \partial_z - g \langle \Gamma \rangle \partial_z = \frac{2}{\bar{\sigma}'} (V_1(z) \mathcal{D}_1 \partial_z + V_0(z) \partial_z),$$

where  $\bar{\sigma}' = \partial_z \bar{\sigma}$  and we introduce the notation

$$V_1(z) = \frac{1}{2} \rho_{xz} \bar{\sigma}'(z) g(z) \langle f(\cdot, z) \rangle, \quad V_0(z) = -\frac{1}{2} \bar{\sigma}'(z) g(z) \langle \Gamma(\cdot, z) \rangle.$$

In order to make use of equation (2.30), we need expressions for  $\langle \mathcal{L}_1 P_{2,1} \rangle$  and  $\langle \mathcal{M}_3 P_{2,0} \rangle$ .

**Proposition 2.2.** *We have the following expressions:*

$$\langle \mathcal{L}_1 P_{2,1} \rangle = (V_3 \mathcal{D}_1 \mathcal{D}_2 + V_2 \mathcal{D}_2) P_{0,1} + \frac{1}{\bar{\sigma}'} (C_2 \mathcal{D}_1^2 + C_1 \mathcal{D}_1 + C_0) \partial_z P_{0,0}, \quad (2.33)$$

$$\langle \mathcal{M}_3 P_{2,0} \rangle = \frac{1}{\bar{\sigma}'} C \mathcal{D}_2 \partial_z P_{0,0}, \quad (2.34)$$

where

$$\begin{aligned} C_2(z) &= -\rho_{xy} \rho_{xz} \bar{\sigma}'(z) g(z) \langle \beta(\cdot) f(\cdot, z) \partial_y \psi_3(\cdot, z) \rangle, \\ C_1(z) &= \rho_{xz} \bar{\sigma}'(z) g(z) \langle \beta(\cdot) \Lambda \partial_y \psi_3 \rangle + \rho_{xy} g \langle \beta(\cdot) f(\cdot, z) \partial_y \psi_4(\cdot, z) \rangle, \\ C_0(z) &= -\bar{\sigma}'(z) g(z) \langle \beta(\cdot) \Lambda(\cdot, z) \partial_y \psi_4(\cdot, z) \rangle, \\ C(z) &= -\frac{1}{2} \rho_{yz} \bar{\sigma}'(z) g(z) \langle \beta(\cdot) \partial_y \phi(\cdot, z) \rangle, \end{aligned}$$

and  $\psi_3(y, z)$  and  $\psi_4(y, z)$  satisfy

$$\mathcal{L}_0 \psi_3 = f - \langle f \rangle, \quad \mathcal{L}_0 \psi_4 = \Gamma - \langle \Gamma \rangle. \quad (2.35)$$

*Proof.* The proof is in Appendix B. □

Inserting (2.33) and (2.34) into (2.30), we find

$$\langle \mathcal{L}_2 \rangle P_{1,1} = -\mathcal{V} P_{0,1} - \frac{1}{\sigma'} \mathcal{C} \partial_z P_{0,0} - \langle \mathcal{M}_1 \rangle P_{1,0}, \quad (2.36)$$

where the  $z$ -dependent operator  $\mathcal{C}$  is given by

$$\mathcal{C}(z) = C_2(z) \mathcal{D}_1^2 + C_1(z) \mathcal{D}_1 + C_0(z) + C(z) \mathcal{D}_2.$$

The solution  $P_{1,1}$  of the PDE (2.36) with terminal condition (2.32) will be given in Proposition 2.3. This is as far as we will take the asymptotic analysis of equation (2.7)

## 2.4 Singular Perturbation Analysis of $\mathcal{O}(\delta)$ Equation (2.8)

We now move on to the  $\mathcal{O}(\delta)$  equation (2.8). Proceeding as in Sections 2.2 and 2.3, we insert expansions (2.5) into (2.8) and collect term of like-powers of  $\sqrt{\varepsilon}$ . The resulting  $\mathcal{O}(\delta/\varepsilon)$  and  $\mathcal{O}(\delta/\sqrt{\varepsilon})$  equations are:

$$\begin{aligned} \mathcal{O}(\delta/\varepsilon) : & \quad 0 = \mathcal{L}_0 P_{0,2}, \\ \mathcal{O}(\delta/\sqrt{\varepsilon}) : & \quad 0 = \mathcal{L}_0 P_{1,2} + \mathcal{L}_1 P_{0,2} + \mathcal{M}_3 \overline{P_{0,1}}, \end{aligned}$$

where we have used  $\mathcal{M}_3 P_{0,1} = 0$  since  $\mathcal{M}_3$  contains  $\partial_y$  and  $P_{0,1}$  is independent of  $y$ . Recalling that all terms in  $\mathcal{L}_0$  and  $\mathcal{L}_1$  also contain  $\partial_y$ , we seek solutions  $P_{0,2}$  and  $P_{1,2}$  of the form

$$P_{0,2} = P_{0,2}(t, x, z), \quad P_{1,2} = P_{1,2}(t, x, z).$$

Continuing the asymptotic analysis, the  $\mathcal{O}(\delta)$  equation is:

$$\mathcal{O}(\delta) : \quad 0 = \mathcal{L}_0 P_{2,2} + \mathcal{L}_1 \overline{P_{1,2}} + \mathcal{L}_2 P_{0,2} + \mathcal{M}_3 \overline{P_{1,1}} + \mathcal{M}_1 P_{0,1} + \mathcal{M}_2 P_{0,0}. \quad (2.37)$$

Equation (2.37) is a Poisson equation of the form (2.12) whose centering condition (2.13) is

$$\mathcal{O}(\delta) : \quad 0 = \langle \mathcal{L}_2 \rangle P_{0,2} + \langle \mathcal{M}_1 \rangle P_{0,1} + \mathcal{M}_2 P_{0,0}. \quad (2.38)$$

We also impose the following terminal condition

$$\mathcal{O}(\delta) : \quad P_{0,2}(T, x, z) = 0, \quad (2.39)$$

The solution  $P_{0,2}$  of the PDE (2.38) with terminal condition (2.39) will be given in Proposition 2.3. This is as far as we will take the combined singular-regular perturbation analysis.

## 2.5 Review of Asymptotic Analysis and Pricing Formulas

In the previous sections we showed (formally) that the price of a European option can be approximated by

$$P^{\varepsilon, \delta} \approx \tilde{P}^{\varepsilon, \delta} := P_{0,0} + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1} + \varepsilon P_{2,0} + \delta P_{2,0} + \sqrt{\varepsilon \delta} P_{1,1}, \quad (2.40)$$

where

$$\left. \begin{aligned} \mathcal{O}(1) : & \quad \langle \mathcal{L}_2 \rangle P_{0,0} = 0, & P_{0,0}(T, x, z) &= h(x), \\ \mathcal{O}(\sqrt{\varepsilon}) : & \quad \langle \mathcal{L}_2 \rangle P_{1,0} = -\mathcal{V} P_{0,0}, & P_{1,0}(T, x, z) &= 0, \\ \mathcal{O}(\sqrt{\delta}) : & \quad \langle \mathcal{L}_2 \rangle P_{0,1} = -\langle \mathcal{M}_1 \rangle P_{0,0}, & P_{0,1}(T, x, z) &= 0, \\ \mathcal{O}(\varepsilon) : & \quad P_{2,0} = -\frac{1}{2} \phi \mathcal{D}_2 P_{0,0} + F_{2,0}, & & \\ & \quad \langle \mathcal{L}_2 \rangle F_{2,0} = -\mathcal{A} P_{0,0} - \mathcal{V} P_{1,0}, & F_{2,0}(T, x, z) &= 0, \\ \mathcal{O}(\delta) : & \quad \langle \mathcal{L}_2 \rangle P_{0,2} = -\langle \mathcal{M}_1 \rangle P_{0,1} - \mathcal{M}_2 P_{0,0}, & P_{0,2}(T, x, z) &= 0, \\ \mathcal{O}(\sqrt{\varepsilon \delta}) : & \quad \langle \mathcal{L}_2 \rangle P_{1,1} = -\mathcal{V} P_{0,1} - \frac{1}{\sigma'} \mathcal{C} \partial_z P_{0,0} - \langle \mathcal{M}_1 \rangle P_{1,0}, & P_{1,1}(T, x, z) &= 0, \end{aligned} \right\} \quad (2.41)$$



and the  $z$ -dependent operators in (2.41) are given by

$$\left. \begin{aligned} \mathcal{O}(1) : \quad \langle \mathcal{L}_2 \rangle &= \partial_t + \frac{1}{2} \bar{\sigma}^2 \mathcal{D}_2 + r \mathcal{D}_1 - r, \\ \mathcal{O}(\sqrt{\varepsilon}) : \quad \mathcal{V} &= V_3 \mathcal{D}_1 \mathcal{D}_2 + V_2 \mathcal{D}_2, \\ \mathcal{O}(\sqrt{\delta}) : \quad \langle \mathcal{M}_1 \rangle &= \frac{2}{\bar{\sigma}'} (V_1 \mathcal{D}_1 + V_0) \partial_z \\ \mathcal{O}(\varepsilon) : \quad \mathcal{A} &= A_2 \mathcal{D}_1^2 \mathcal{D}_2 + A_1 \mathcal{D}_1 \mathcal{D}_2 + A_0 \mathcal{D}_2 + A \mathcal{D}_2^2, \\ \mathcal{O}(\delta) : \quad \mathcal{M}_2 &= \frac{1}{2} g^2 \partial_{zz}^2 + c \partial_z, \\ \mathcal{O}(\sqrt{\varepsilon \delta}) : \quad \mathcal{C} &= C_2 \mathcal{D}_1^2 + C_1 \mathcal{D}_1 + C_0 + C \mathcal{D}_2. \end{aligned} \right\} \quad (2.42)$$

In the following, we provide explicit expressions for  $\{P_{i,j}, i+j \leq 2\}$ .

**Proposition 2.3.** *Introducing  $\tau = T - t$ , we have the following expressions for the  $\{P_{i,j}\}$ :*

$$\begin{aligned} P_{0,0} &= P_{BS}(\bar{\sigma}(z)), \quad P_{1,0} = \tau \mathcal{V} P_{BS}(\bar{\sigma}(z)), \quad P_{0,1} = \tau \mathcal{N}_1 \partial_\sigma P_{BS}(\bar{\sigma}(z)) \\ P_{2,0} &= -\frac{1}{2} \phi \mathcal{D}_2 P_{BS}(\bar{\sigma}(z)) + F_{2,0}, \quad \text{where } F_{2,0} = \left( \tau \mathcal{A} + \frac{1}{2} \tau^2 \mathcal{V}^2 \right) P_{BS}(\bar{\sigma}(z)), \\ P_{0,2} &= \left( \frac{2\tau^2}{3\bar{\sigma}'} \mathcal{N}_1 \mathcal{N}_1' \partial_\sigma + \frac{\tau^2}{2} \mathcal{N}_1^2 \left( \partial_{\sigma\sigma}^2 + \frac{1}{3\bar{\sigma}} \partial_\sigma \right) + \frac{\tau}{3} B_2 \left( \partial_{\sigma\sigma}^2 + \frac{1}{2\bar{\sigma}} \partial_\sigma \right) + \frac{\tau}{2} B_1 \partial_\sigma \right) P_{BS}(\bar{\sigma}(z)), \\ P_{1,1} &= \left( \tau^2 \mathcal{V} \mathcal{N}_1 \partial_\sigma + \frac{\tau}{2} \mathcal{C} \partial_\sigma + \frac{\tau^2}{\bar{\sigma}'} \mathcal{N}_1 \mathcal{V}' \right) P_{BS}(\bar{\sigma}(z)), \end{aligned}$$

where we have introduced the  $z$ -dependent operators

$$\mathcal{N}_1 = V_1 \mathcal{D}_1 + V_0, \quad \mathcal{N}_1' = V_1' \mathcal{D}_1 + V_0', \quad \mathcal{V}' = V_3' \mathcal{D}_1 \mathcal{D}_2 + V_2' \mathcal{D}_2,$$

and parameters

$$V_j' = \partial_z V_j, \quad j = 0, 1, 2, 3 \quad B_2 = \frac{1}{2} g^2 (\bar{\sigma}')^2, \quad B_1 = \frac{1}{2} g^2 \bar{\sigma}'' + c \bar{\sigma}'.$$

*Proof.* The proof is in Appendix C. □

## 2.6 Accuracy of the Approximation

To establish the accuracy of our pricing approximation  $\tilde{P}^{\varepsilon, \delta}$  defined in (2.40), we make the following assumptions:

1. The system of SDEs (2.1) has a unique strong solution  $(X, Y, Z)$  for fixed  $\varepsilon, \delta \leq 1$ .
2. The market prices of volatility risk are bounded:  $\|\Lambda\|_\infty < \infty$  and  $\|\Gamma\|_\infty < \infty$ .
3. Let  $Y^{(1)}$  be a diffusion process whose infinitesimal generator is  $\mathcal{L}_0$  (so that, in distribution,  $Y_t = Y_{t/\varepsilon}^{(1)}$  under  $\mathbb{P}$ ). We assume that  $Y^{(1)}$  is ergodic and has a unique invariant distribution  $\Pi$  with density  $\pi$ , and that  $\mathcal{L}_0$  has a positive spectral gap. We note that two of the processes that are most commonly used to model volatility — the Cox-Ingersoll-Ross (CIR) and Ornstein-Uhlenbeck (OU) processes — satisfy these assumptions.
4. The process  $Y^{(1)}$  admits moments of any order uniformly bounded in  $t$ :

$$\sup_t \mathbb{E} \left[ \left| Y_t^{(1)} \right|^k \right] \leq C(k).$$

5. Let  $Z^{(1)}$  be a diffusion process whose infinitesimal generator is  $\mathcal{M}_2$  (so that, in distribution,  $Z_t = Z_{\delta t}^{(1)}$  under  $\mathbb{P}$ ). We assume that  $Z^{(1)}$  admits moments of any order uniformly bounded in  $t < T$ :

$$\sup_{t \leq T} \mathbb{E} \left[ \left| Z_t^{(1)} \right|^k \right] \leq C(T, k).$$

6. We assume that the function  $f(y, z)$  is smooth in  $z$ , that the solutions  $\phi(y, z)$  and  $\{\psi_i(y, z), i \leq 4\}$  to equations (2.20), (2.26) and (2.35), are at most polynomially growing in  $y$  and  $z$ , and that

$$\bar{\sigma}^2(z) := \int f^2(y, z) \Pi(dy) < \infty.$$

7. We assume that the payoff function  $h(x)$  is smooth and bounded with bounded derivatives. The proof of accuracy provided here (in Appendix D) uses this smoothness assumption. It can be generalized to nonsmooth payoff functions (such as in the case of call options considered in Section 3) following the regularization argument given in [6] for the first order approximation with a fast factor. The details of the generalization to the second order approximation with fast and slow factors are quite lengthy and we omit them here.

**Theorem 2.4.** *For fixed  $t < T$ ,  $x$ ,  $y$ , and  $z$ , the model price  $P^{\varepsilon, \delta}$  solution of (2.2) and our price approximation  $\tilde{P}^{\varepsilon, \delta}$  defined by (2.40) satisfy*

$$|P^{\varepsilon, \delta}(t, x, y, z) - \tilde{P}^{\varepsilon, \delta}(t, x, y, z)| = \mathcal{O}(\varepsilon^{3/2} + \varepsilon\sqrt{\delta} + \delta\sqrt{\varepsilon}).$$

*Proof.* The proof is given in Appendix D. □

**Remark** (Terminal Layer Analysis). The main difficulty in extending the accuracy of our pricing approximation from first order to second order is the treatment of the terminal condition for the second order term  $P_{2,0}$  arising from the singular expansion due to the fast factor  $Y$ . In [16], the solution  $P_{2,0}$  is derived by a formal matched asymptotic expansion with a terminal layer of size  $\varepsilon$ . Here, in Appendix D, we provide a probabilistic proof based on the ergodic property of the fast factor  $Y$ , which justifies the choice of terminal condition made in (2.22).

## 2.7 Group Parameters and Exotic Option Pricing

We now summarize the parameters needed in the pricing approximation formulas derived in the previous section. We begin by separating the  $y$ -dependent part in  $\tilde{P}^{\varepsilon, \delta}$  given by (2.40), by writing

$$\tilde{P}^{\varepsilon, \delta}(t, x, y, z) = -\frac{1}{2} \varepsilon \phi(y, z) \mathcal{D}_2 P_{0,0}(t, x, z) + \tilde{Q}^{\varepsilon, \delta}(t, x, z),$$

where

$$\tilde{Q}^{\varepsilon, \delta}(t, x, z) := P_{0,0} + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1} + \sqrt{\varepsilon\delta} P_{1,1} + \varepsilon F_{2,0} + \delta P_{0,2}. \quad (2.43)$$

Using (2.40), (2.41) and the linearity of the operator  $\langle \mathcal{L}_2 \rangle$ , we find that  $\tilde{Q}^{\varepsilon, \delta}$  satisfies the following PDE and terminal condition

$$\langle \mathcal{L}_2 \rangle \tilde{Q}^{\varepsilon, \delta} = S^{\varepsilon, \delta}, \quad \tilde{Q}^{\varepsilon, \delta}(T, x, z) = h(x),$$

where the source term  $S^{\varepsilon, \delta}$  is given by

$$\begin{aligned} S^{\varepsilon, \delta} &= -\sqrt{\varepsilon} \mathcal{V} P_{0,0} - \sqrt{\delta} \langle \mathcal{M}_1 \rangle P_{0,0} - \sqrt{\varepsilon\delta} \left( \mathcal{V} P_{0,1} + \frac{1}{\sigma'} \mathcal{C} \partial_z P_{0,0} + \langle \mathcal{M}_1 \rangle P_{1,0} \right) \\ &\quad - \varepsilon \left( \mathcal{A} P_{0,0} + \mathcal{V} P_{1,0} \right) - \delta \left( \langle \mathcal{M}_1 \rangle P_{0,1} + \mathcal{M}_2 P_{0,0} \right) \\ &= -(\sqrt{\varepsilon} \mathcal{V}) P_{0,0} - (\sqrt{\delta} \langle \mathcal{M}_1 \rangle) P_{0,0} - (\sqrt{\varepsilon} \mathcal{V})(\sqrt{\delta} P_{0,1}) - (\sqrt{\varepsilon\delta} \mathcal{C}) \frac{1}{\sigma'} \partial_z P_{0,0} - (\sqrt{\delta} \langle \mathcal{M}_1 \rangle)(\sqrt{\varepsilon} P_{1,0}) \\ &\quad - (\varepsilon \mathcal{A}) P_{0,0} - (\sqrt{\varepsilon} \mathcal{V})(\sqrt{\varepsilon} P_{1,0}) - (\sqrt{\delta} \langle \mathcal{M}_1 \rangle)(\sqrt{\delta} P_{0,1}) - (\delta \mathcal{M}_2) P_{0,0}. \end{aligned}$$

To extract which group parameters are needed for the price expansion, we absorb a half-integer power of  $\varepsilon$  and/or  $\delta$  into the corresponding group parameters and define:

$$V_i^\varepsilon := \sqrt{\varepsilon} V_i, \quad V_i^\delta := \sqrt{\delta} V_i, \quad A_i^\varepsilon := \varepsilon A_i, \quad B_i^\delta := \delta B_i, \quad C_i^{\varepsilon,\delta} := \sqrt{\varepsilon\delta} C_i. \quad (2.44)$$

Similarly, we absorb the appropriate  $\varepsilon$  or  $\delta$  pre-multiplier into the terms of the expansion (2.43) by defining  $P_{1,0}^\varepsilon$  and  $P_{0,1}^\delta$  through

$$\sqrt{\varepsilon} P_{1,0}(t, x, z) = P_{1,0}^\varepsilon(t, x; \bar{\sigma}(z), V_2^\varepsilon(z), V_3^\varepsilon(z)), \quad \sqrt{\delta} P_{0,1}(t, x, z) = P_{0,1}^\delta(t, x; \bar{\sigma}(z), V_0^\delta(z), V_1^\delta(z)).$$

Substituting from (2.42) the expressions for  $\mathcal{M}_2, \mathcal{V}, \mathcal{A}, \langle \mathcal{M}_1 \rangle$  and  $\mathcal{C}$ , and changing the  $\partial_z$  derivatives in  $\langle \mathcal{M}_1 \rangle$  and  $\mathcal{M}_2$  acting on  $P_{0,0}$  into  $\partial_\sigma$  derivatives acting on  $P_{BS}(\bar{\sigma}(z))$ , we finally have

$$\begin{aligned} S^{\varepsilon,\delta} = & - (V_3^\varepsilon \mathcal{D}_1 \mathcal{D}_2 + V_2^\varepsilon \mathcal{D}_2) P_{BS} - 2 (V_1^\delta \mathcal{D}_1 + V_0^\delta) \partial_\sigma P_{BS} \\ & - (V_3^\varepsilon \mathcal{D}_1 \mathcal{D}_2 + V_2^\varepsilon \mathcal{D}_2) P_{0,1}^\delta - \left( C_2^{\varepsilon,\delta} \mathcal{D}_1^2 + C_1^{\varepsilon,\delta} \mathcal{D}_1 + C_0^{\varepsilon,\delta} + C^{\varepsilon,\delta} \mathcal{D}_2 \right) \partial_\sigma P_{BS} \\ & - 2 (V_1^\delta \mathcal{D}_1 + V_0^\delta) \left( \partial_\sigma + \frac{V_3^{\prime\varepsilon}}{\bar{\sigma}'} \partial_{V_3^\varepsilon} + \frac{V_2^{\prime\varepsilon}}{\bar{\sigma}'} \partial_{V_2^\varepsilon} \right) P_{1,0}^\varepsilon \\ & - \left( A_2^\varepsilon \mathcal{D}_1^2 \mathcal{D}_2 + A_1^\varepsilon \mathcal{D}_1 \mathcal{D}_2 + A_0^\varepsilon \mathcal{D}_2 + A^\varepsilon \mathcal{D}_2^2 \right) P_{BS} - (V_3^\varepsilon \mathcal{D}_1 \mathcal{D}_2 + V_2^\varepsilon \mathcal{D}_2) P_{1,0}^\varepsilon \\ & - 2 (V_1^\delta \mathcal{D}_1 + V_0^\delta) \left( \partial_\sigma + \frac{V_1^{\prime\delta}}{\bar{\sigma}'} \partial_{V_1^\delta} + \frac{V_0^{\prime\delta}}{\bar{\sigma}'} \partial_{V_0^\delta} \right) P_{0,1}^\delta - (B_2^\delta \partial_{\sigma\sigma}^2 + B_1^\delta \partial_\sigma) P_{BS}. \end{aligned}$$

Here our notation is  $V_i^{\prime\varepsilon}(z) = \partial_z V_i^\varepsilon(z)$ , and similarly  $V_i^{\prime\delta}$ . Since  $P_{1,0}$  is linear in  $V_3$  and  $V_2$  and  $P_{0,1}$  is linear in  $V_1$  and  $V_0$ , neither  $\partial_{V_3^\varepsilon} P_{1,0}$ ,  $\partial_{V_2^\varepsilon} P_{1,0}$ ,  $\partial_{V_1^\delta} P_{0,1}$  nor  $\partial_{V_0^\delta} P_{0,1}$  contain any of the  $V_i$ 's (that is, they are order one quantities).

As such, the *group parameters* that appear in the source term  $S^{\varepsilon,\delta}$  and therefore, in the price approximation (2.40) are

$$V_3^\varepsilon, V_2^\varepsilon, V_1^\delta, V_0^\delta, C_2^{\varepsilon,\delta}, C_1^{\varepsilon,\delta}, C_0^{\varepsilon,\delta}, C^{\varepsilon,\delta}, A_2^\varepsilon, A_1^\varepsilon, A_0^\varepsilon, A^\varepsilon, B_2^\delta, B_1^\delta, \frac{V_3^{\prime\varepsilon}}{\bar{\sigma}'}, \frac{V_2^{\prime\varepsilon}}{\bar{\sigma}'}, \frac{V_1^{\prime\delta}}{\bar{\sigma}'}, \frac{V_0^{\prime\delta}}{\bar{\sigma}'}. \quad (2.45)$$

These 18 parameters, which move with the slow volatility factor  $Z_t$ , as well as  $\phi^\varepsilon(y, z) := \varepsilon \phi(y, z)$  needed in (2.40), can be obtained by calibrating the class of multiscale stochastic volatility models to the implied volatility surface of (liquid) European options, as described the Section 3.2. Note from (2.44) that the  $V_i^\varepsilon$  are order  $\sqrt{\varepsilon}$ , the  $V_i^\delta$  order  $\sqrt{\delta}$  and that they appeared in the first order asymptotic theory in [7]. The new parameters ( $A_i^\varepsilon, B_i^\delta, C_i^{\varepsilon,\delta}$ ) come from the order  $\varepsilon$ , order  $\delta$  and order  $\sqrt{\varepsilon\delta}$  terms in the the second order expansion respectively.

### 2.7.1 Parameter Reduction

The group parameters in (2.45) depend on the current level  $z$  of the slow volatility factor and, in the case of  $\phi^\varepsilon$ , on the fast factor too. In order to calibrate completely from the implied volatility surface and not use historical returns data to estimate  $\bar{\sigma}(z)$ , we replace it by a quantity  $\sigma^*(z)$  which absorbs the term  $V_2^\varepsilon(z)$ . In so doing, there is now one less parameter, and we show that the accuracy of the second order approximation is unchanged. The accuracy of our approximation is given in Theorem 2.4 and the following Proposition refers to it.

**Proposition 2.5** (Parameter Reduction). *Without loss of accuracy, in the pricing equations and formulas obtained in this section, we can replace  $\bar{\sigma}(z)$  by  $\sigma^*(z)$  defined by*

$$\sigma^*(z) := \sqrt{\bar{\sigma}(z)^2 + 2V_2^\varepsilon(z)}, \quad (2.46)$$

*replace  $P_{0,0}$  by the Black-Scholes price at volatility  $\sigma^*(z)$ , and remove the  $V_2$ -dependent terms.*

*Proof.* The proof is in appendix E. □

### 2.7.2 Pricing Exotic Options

It is important to remember that the formulas given for the  $P_{i,j}$ 's in Theorem 2.3 are valid for *European* options only. This is because Theorem 2.3 relies on the Vega-Gamma relationship (C.1), which does not hold in general for exotic options. Nevertheless, the PDEs derived in (2.41) *are* valid for many path-dependent options with appropriate boundary conditions (e.g. barrier options). The approximate price of a path-dependent option can be found by numerically solving an inhomogeneous PDE (with additional boundary conditions), the main point being that the parameters involved are the same as those appearing in European option prices used for calibration as described in Section 3.2. A full development for particular contracts is outside the scope of this paper.

## 3 Asymptotics for Implied Volatilities and Calibration

In what follows, we give the second order expansion of Black-Scholes implied volatility, which is obtained by inverting the Black-Scholes formula for European call options with respect to the volatility parameter.

### 3.1 Implied Volatility Expansion

In practice, it is common to find the parameters of a pricing model by calibrating the model to the implied volatility surface of liquid European calls and puts, rather than by calibrating to call and put prices directly. To this end, we seek an implied volatility expansion

$$I^{\varepsilon,\delta} = \sum_{j \geq 0} \sum_{i \geq 0} \sqrt{\varepsilon}^i \sqrt{\delta}^j I_{i,j} \quad \text{such that} \quad P^{\varepsilon,\delta} = P_{BS}(I^{\varepsilon,\delta}).$$

Performing a Taylor expansion of  $P_{BS}(I^{\varepsilon,\delta})$  about  $I_{0,0}$  and rearranging terms yields

$$\begin{aligned} & P_{0,0} + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1} + \sqrt{\varepsilon\delta} P_{1,1} + \varepsilon P_{2,0} + \delta P_{0,2} + \dots \\ &= P_{BS}(I_{0,0} + \sqrt{\varepsilon} I_{1,0} + \sqrt{\delta} I_{0,1} + \sqrt{\varepsilon\delta} I_{1,1} + \varepsilon I_{2,0} + \delta I_{0,2} + \dots) \\ &= P_{BS}(I_{0,0}) + \left( \sqrt{\varepsilon} I_{1,0} + \sqrt{\delta} I_{0,1} + \dots \right) \partial_{\sigma} P_{BS}(I_{0,0}) \\ &\quad + \frac{1}{2} \left( \sqrt{\varepsilon} I_{1,0} + \sqrt{\delta} I_{0,1} + \dots \right)^2 \partial_{\sigma\sigma}^2 P_{BS}(I_{0,0}) + \dots \\ &= P_{BS}(I_{0,0}) + \sqrt{\varepsilon} I_{1,0} \partial_{\sigma} P_{BS}(I_{0,0}) + \sqrt{\delta} I_{0,1} \partial_{\sigma} P_{BS}(I_{0,0}) \\ &\quad + \sqrt{\varepsilon\delta} \left( I_{1,0} I_{0,1} \partial_{\sigma\sigma}^2 P_{BS}(I_{0,0}) + I_{1,1} \partial_{\sigma} P_{BS}(I_{0,0}) \right) \\ &\quad + \varepsilon \left( \frac{1}{2} I_{1,0}^2 \partial_{\sigma\sigma}^2 P_{BS}(I_{0,0}) + I_{2,0} \partial_{\sigma} P_{BS}(I_{0,0}) \right) \\ &\quad + \delta \left( \frac{1}{2} I_{0,1}^2 \partial_{\sigma\sigma}^2 P_{BS}(I_{0,0}) + I_{0,2} \partial_{\sigma} P_{BS}(I_{0,0}) \right) + \dots \end{aligned} \tag{3.1}$$

Equating terms in (3.1) of like powers of  $\sqrt{\varepsilon}$  and  $\sqrt{\delta}$ , and using  $P_{0,0} = P_{BS}(\bar{\sigma})$  we find

$$\left. \begin{aligned} \mathcal{O}(1): \quad I_{0,0} &= \bar{\sigma}, & \mathcal{O}(\varepsilon): \quad I_{2,0} &= \frac{P_{2,0}}{\partial_{\sigma} P_{0,0}} - \frac{1}{2} I_{1,0}^2 \frac{\partial_{\sigma\sigma}^2 P_{0,0}}{\partial_{\sigma} P_{0,0}}, \\ \mathcal{O}(\sqrt{\varepsilon}): \quad I_{1,0} &= \frac{P_{1,0}}{\partial_{\sigma} P_{0,0}}, & \mathcal{O}(\delta): \quad I_{0,2} &= \frac{P_{0,2}}{\partial_{\sigma} P_{0,0}} - \frac{1}{2} I_{0,1}^2 \frac{\partial_{\sigma\sigma}^2 P_{0,0}}{\partial_{\sigma} P_{0,0}}, \\ \mathcal{O}(\sqrt{\delta}): \quad I_{0,1} &= \frac{P_{0,1}}{\partial_{\sigma} P_{0,0}}, & \mathcal{O}(\sqrt{\varepsilon\delta}): \quad I_{1,1} &= \frac{P_{1,1}}{\partial_{\sigma} P_{0,0}} - I_{1,0} I_{0,1} \frac{\partial_{\sigma\sigma}^2 P_{0,0}}{\partial_{\sigma} P_{0,0}}. \end{aligned} \right\} \tag{3.2}$$

For a European call or put option with strike price  $K$  and time to maturity  $\tau$  it is convenient to express the  $I_{i,j}$ 's as functions of *forward log-moneyness*

$$d := \log(K/x e^{r\tau}) \quad (\text{forward log-moneyness}).$$

Setting the payoff function  $h(x) = (x - K)^+$  for a call option and using the expressions given for  $\{P_{i,j}\}$  in Theorem 2.3, the  $I_{i,j}$ 's in (3.2) become

$$\left. \begin{aligned} \mathcal{O}(1) : \quad I_{0,0} &= \bar{\sigma}, \\ \mathcal{O}(\sqrt{\varepsilon}) : \quad I_{1,0} &= V_2 \frac{1}{\bar{\sigma}} + V_3 \left( \frac{1}{2\bar{\sigma}} + \frac{d}{\tau \bar{\sigma}^3} \right), \\ \mathcal{O}(\sqrt{\delta}) : \quad I_{0,1} &= V_0 \tau + V_1 \left( \frac{\tau}{2} + \frac{d}{\bar{\sigma}^2} \right), \\ \mathcal{O}(\varepsilon) : \quad I_{2,0} &= \frac{-\phi}{2\tau \bar{\sigma}} + V_2^2 \left( -\frac{1}{2\bar{\sigma}^3} \right) + V_2 V_3 \left( -\frac{3d}{\tau \bar{\sigma}^5} - \frac{1}{2\bar{\sigma}^3} \right) \\ &\quad + V_3^2 \left( -\frac{3d^2}{\tau^2 \bar{\sigma}^7} + \frac{3}{2\tau \bar{\sigma}^5} - \frac{3d}{2\tau \bar{\sigma}^5} \right) \\ &\quad + A \left( \frac{d^2}{\tau^2 \bar{\sigma}^5} - \frac{1}{\tau \bar{\sigma}^3} - \frac{1}{4\bar{\sigma}} \right) + A_0 \left( \frac{1}{\bar{\sigma}} \right) + A_1 \left( \frac{d}{\tau \bar{\sigma}^3} + \frac{1}{2\bar{\sigma}} \right) \\ &\quad + A_2 \left( \frac{d^2}{\tau^2 \bar{\sigma}^5} - \frac{1}{\tau \bar{\sigma}^3} + \frac{d}{\tau \bar{\sigma}^3} + \frac{1}{4\bar{\sigma}} \right), \\ \mathcal{O}(\delta) : \quad I_{0,2} &= V_0^2 \left( \frac{\tau^2}{6\bar{\sigma}} \right) + V_0 V_1 \left( -\frac{5d\tau}{3\bar{\sigma}^3} + \frac{\tau^2}{6\bar{\sigma}} \right) + V_1^2 \left( -\frac{7d^2}{3\bar{\sigma}^5} + \frac{5\tau}{6\bar{\sigma}^3} - \frac{5d\tau}{6\bar{\sigma}^3} + \frac{\tau^2}{6\bar{\sigma}} \right) \\ &\quad + V_0 \frac{V_0'}{\bar{\sigma}'} \left( \frac{2\tau^2}{3} \right) + V_0 \frac{V_1'}{\bar{\sigma}'} \left( \frac{\tau^2}{3} + \frac{2d\tau}{3\bar{\sigma}^2} \right) + V_1 \frac{V_0'}{\bar{\sigma}'} \left( \frac{\tau^2}{3} + \frac{2d\tau}{3\bar{\sigma}^2} \right) \\ &\quad + V_1 \frac{V_1'}{\bar{\sigma}'} \left( \frac{\tau^2}{6} + \frac{2d^2}{3\bar{\sigma}^4} - \frac{2\tau}{3\bar{\sigma}^2} + \frac{2d\tau}{3\bar{\sigma}^2} \right) + B_2 \left( \frac{d^2}{3\bar{\sigma}^3} + \frac{\tau}{6\bar{\sigma}} - \frac{\tau^2 \bar{\sigma}}{12} \right) + B_1 \left( \frac{\tau}{2} \right), \\ \mathcal{O}(\sqrt{\varepsilon}\delta) : \quad I_{1,1} &= V_0 V_2 \left( -\frac{\tau}{\bar{\sigma}^2} \right) + V_0 V_3 \left( -\frac{3d}{\bar{\sigma}^4} - \frac{\tau}{2\bar{\sigma}^2} \right) \\ &\quad + V_1 V_2 \left( -\frac{3d}{\bar{\sigma}^4} - \frac{\tau}{2\bar{\sigma}^2} \right) + V_1 V_3 \left( -\frac{6d^2}{\tau \bar{\sigma}^6} + \frac{3}{\bar{\sigma}^4} - \frac{3d}{\bar{\sigma}^4} \right) \\ &\quad + V_0 \frac{V_2'}{\bar{\sigma}'} \left( \frac{\tau}{\bar{\sigma}} \right) + V_0 \frac{V_3'}{\bar{\sigma}'} \left( \frac{d}{\bar{\sigma}^3} + \frac{\tau}{2\bar{\sigma}} \right) + V_1 \frac{V_2'}{\bar{\sigma}'} \left( \frac{d}{\bar{\sigma}^3} + \frac{\tau}{2\bar{\sigma}} \right) \\ &\quad + V_1 \frac{V_3'}{\bar{\sigma}'} \left( \frac{d^2}{\tau \bar{\sigma}^5} - \frac{1}{\bar{\sigma}^3} + \frac{d}{\bar{\sigma}^3} + \frac{\tau}{4\bar{\sigma}} \right) + C_2 \left( \frac{\tau}{8} + \frac{d^2}{2\tau \bar{\sigma}^4} - \frac{1}{2\bar{\sigma}^2} + \frac{d}{2\bar{\sigma}^2} \right) \\ &\quad + C_1 \left( \frac{\tau}{4} + \frac{d}{2\bar{\sigma}^2} \right) + C_0 \left( \frac{\tau}{2} \right) + C \left( -\frac{\tau}{8} + \frac{d^2}{2\tau \bar{\sigma}^4} - \frac{1}{2\bar{\sigma}^2} \right). \end{aligned} \right\} \quad (3.3)$$

Observe that this second order expansion produces an implied volatility curve which is *quadratic* in log-moneyness  $d$  and therefore accounts for the slight turn in the skew that is most prominent in shorter maturity options data, as we will see in Figure 1. The first order approximation derived in [7] is linear in  $d$  and therefore only accounted for the skew effect. Note also that the parameter reduction outlined in Proposition 2.5 can be applied to this implied volatility expansion as well ( $\bar{\sigma}$  replaced by  $\sigma^*$  and  $V_2$ -terms removed), and this will be used in the calibration in the next section. We also remark that the formal second order expansion for the case of a *single slow* volatility factor had previously been considered in [9], [18] and [20], for instance.

### 3.2 Calibration

In this section we discuss how the parameters (2.45), which are needed to price exotic options as discussed at the end of Section 2.7, can be obtained by calibrating the multiscale class of models to liquid European option data. We define

$$\tilde{I}^{\varepsilon,\delta} := I_{0,0} + \sqrt{\varepsilon} I_{1,0} + \sqrt{\delta} I_{0,1} + \sqrt{\varepsilon\delta} I_{1,1} + \varepsilon I_{2,0} + \delta I_{0,2}.$$

Using (3.3) and the parameter reduction described in Proposition 2.5, we have

$$\tilde{I}^{\varepsilon,\delta} = \left( \frac{1}{\tau} k + l + \tau m + \tau^2 n \right) + \frac{d}{\tau} (p + \tau q + \tau^2 s) + \frac{d^2}{\tau^2} (u + \tau v + \tau^2 w), \quad (3.4)$$

where

$$\left. \begin{aligned} \mathcal{O}(1/\tau) : \quad k &= \frac{3(V_3^\varepsilon)^2}{2(\sigma^*)^5} - \frac{A_2^\varepsilon}{(\sigma^*)^3} - \frac{A^\varepsilon}{(\sigma^*)^3} - \frac{\phi^\varepsilon}{2\sigma^*}, \\ \mathcal{O}(1) : \quad l &= \frac{3V_1^\delta V_3^\varepsilon}{(\sigma^*)^4} - \frac{C_2^{\varepsilon,\delta}}{2(\sigma^*)^2} - \frac{C^{\varepsilon,\delta}}{2(\sigma^*)^2} \\ &\quad + \frac{A_0^\varepsilon}{\sigma^*} + \frac{A_1^\varepsilon}{2\sigma^*} + \frac{A_2^\varepsilon}{4\sigma^*} - \frac{A^\varepsilon}{4\sigma^*} - \frac{V_1^\delta V_3^{\prime\varepsilon}}{(\sigma^*)^3 \sigma^{*\prime}} + \sigma^* + \frac{V_3^\varepsilon}{2\sigma^*}, \\ \mathcal{O}(\tau) : \quad m &= \frac{B_1^\delta}{2} + \frac{C_0^{\varepsilon,\delta}}{2} + \frac{C_1^{\varepsilon,\delta}}{4} + \frac{C_2^{\varepsilon,\delta}}{8} - \frac{C^{\varepsilon,\delta}}{8} + \frac{5(V_1^\delta)^2}{6(\sigma^*)^3} \\ &\quad - \frac{V_0^\delta V_3^\varepsilon}{2(\sigma^*)^2} + \frac{B_2^\delta}{6\sigma^*} - \frac{2V_1^\delta V_1^{\prime\delta}}{3(\sigma^*)^2 \sigma^{*\prime}} + \frac{V_0^\delta V_3^{\prime\varepsilon}}{2\sigma^* \sigma^{*\prime}} + \frac{V_1^\delta V_3^{\prime\varepsilon}}{4\sigma^* \sigma^{*\prime}} + V_0^\delta + \frac{V_1^\delta}{2}, \\ \mathcal{O}(\tau^2) : \quad n &= \frac{(V_0^\delta)^2}{6\sigma^*} + \frac{V_0^\delta V_1^\delta}{6\sigma^*} + \frac{(V_1^\delta)^2}{6\sigma^*} - \frac{B_2^\delta \sigma^*}{12} + \frac{2V_0^\delta V_0^{\prime\delta}}{3\sigma^{*\prime}} \\ &\quad + \frac{V_0^{\prime\delta} V_1^\delta}{3\sigma^{*\prime}} + \frac{V_0^\delta V_1^{\prime\delta}}{3\sigma^{*\prime}} + \frac{V_1^\delta V_1^{\prime\delta}}{6\sigma^{*\prime}}, \\ \mathcal{O}(d/\tau) : \quad p &= -\frac{3(V_3^\varepsilon)^2}{2(\sigma^*)^5} + \frac{A_1^\varepsilon}{(\sigma^*)^3} + \frac{A_2^\varepsilon}{(\sigma^*)^3} + \frac{V_3^\varepsilon}{(\sigma^*)^3}, \\ \mathcal{O}(d) : \quad q &= -\frac{3V_0^\delta V_3^\varepsilon}{(\sigma^*)^4} - \frac{3V_1^\delta V_3^\varepsilon}{(\sigma^*)^4} + \frac{C_1^{\varepsilon,\delta}}{2(\sigma^*)^2} + \frac{C_2^{\varepsilon,\delta}}{2(\sigma^*)^2} \\ &\quad + \frac{V_0^\delta V_3^{\prime\varepsilon}}{(\sigma^*)^3 \sigma^{*\prime}} + \frac{V_1^\delta V_3^{\prime\varepsilon}}{(\sigma^*)^3 \sigma^{*\prime}} + \frac{V_1^\delta}{(\sigma^*)^2}, \\ \mathcal{O}(d\tau) : \quad s &= -\frac{5V_0^\delta V_1^\delta}{3(\sigma^*)^3} - \frac{5(V_1^\delta)^2}{6(\sigma^*)^3} + \frac{2V_0^{\prime\delta} V_1^\delta}{3(\sigma^*)^2 \sigma^{*\prime}} + \frac{2V_0^\delta V_1^{\prime\delta}}{3(\sigma^*)^2 \sigma^{*\prime}} + \frac{2V_1^\delta V_1^{\prime\delta}}{3(\sigma^*)^2 \sigma^{*\prime}}, \\ \mathcal{O}(d^2/\tau^2) : \quad u &= -\frac{3(V_3^\varepsilon)^2}{(\sigma^*)^7} + \frac{A_2^\varepsilon}{(\sigma^*)^5} + \frac{A^\varepsilon}{(\sigma^*)^5}, \\ \mathcal{O}(d^2/\tau) : \quad v &= -\frac{6V_1^\delta V_3^\varepsilon}{(\sigma^*)^6} + \frac{C_2^{\varepsilon,\delta}}{2(\sigma^*)^4} + \frac{C^{\varepsilon,\delta}}{2(\sigma^*)^4} + \frac{V_1^\delta V_3^{\prime\varepsilon}}{(\sigma^*)^5 \sigma^{*\prime}}, \\ \mathcal{O}(d^2) : \quad w &= -\frac{7(V_1^\delta)^2}{3(\sigma^*)^5} + \frac{B_2^\delta}{3(\sigma^*)^3} + \frac{2V_1^\delta V_1^{\prime\delta}}{3(\sigma^*)^4 \sigma^{*\prime}}. \end{aligned} \right\} \quad (3.5)$$

In total, we have ten “basis functions” with which to fit the empirically observed implied volatility surface:

$$\left\{ \frac{1}{\tau}, 1, \tau, \tau^2, \frac{d}{\tau}, d, d\tau, \frac{d^2}{\tau^2}, \frac{d^2}{\tau}, d^2 \right\}.$$

It will be helpful to define

$$\Theta := \{k, l, m, n, p, q, s, u, v, w\},$$

$$\Phi := \{\sigma^*, V_3^\varepsilon, V_1^\delta, V_0^\delta, C_2^{\varepsilon, \delta}, C_1^{\varepsilon, \delta}, C_0^{\varepsilon, \delta}, C^{\varepsilon, \delta}, A_2^\varepsilon, A_1^\varepsilon, A_0^\varepsilon, A^\varepsilon, B_2^\delta, B_1^\delta, \frac{V_3'^\varepsilon}{\sigma'}, \frac{V_1'^\delta}{\sigma'}, \frac{V_0'^\delta}{\sigma'}, \phi^\varepsilon\}.$$

We let  $I(\tau, d)$  be the implied volatility of a European call option with time to maturity  $\tau$  and forward log-moneyness  $d$  as observed from option prices on the market. We let  $\widehat{I}^{\varepsilon, \delta}(\tau, d; \Theta)$  be the implied volatility of a European call as calculated using (3.4). The calibration procedure consists of the following steps:

1. Find  $\Theta^*$  such that

$$\min_{\Theta} \sum_i \sum_j \left( I(\tau_i, d_j) - \widehat{I}^{\varepsilon, \delta}(\tau_i, d_j; \Theta) \right)^2 = \sum_i \sum_j \left( I(\tau_i, d_j) - \widehat{I}^{\varepsilon, \delta}(\tau_i, d_j; \Theta^*) \right)^2,$$

where the double sum runs over all maturities  $\tau_i$  and strikes  $K_j$  (corresponding to forward log-moneyness  $d_j$ ) for which a call or put is liquidly traded. This is the least-squares fit of formula (3.4) resulting in estimated  $k, l, m, \dots, w$ .

2. Next the ten constraints of equation (3.5) are used to find the minimal  $L_2$  set of parameters  $\Phi^*$ . That is, we find  $\Phi^*$  such that

$$\min_{\Phi \in \mathcal{J}} \|\Phi\|^2 = \|\Phi^*\|^2, \quad \mathcal{J} = \{\Phi : \text{equation (3.5) holds with } \Theta = \Theta^*\}.$$

We emphasize that our calibration procedure encompasses all maturities, that is we do not fit maturity-by-maturity.

### 3.3 Data

We perform the described calibration procedure on European call and put options on the S&P500 index on two separate dates, one pre-crisis on October 19, 2006, and one post-crisis on March 18, 2010. In Figure 1 we plot the implied volatility fit from October 19, 2006. The parameters obtained from the above calibration procedure are

$$\begin{aligned} \sigma^* &= 0.2051, & V_3^\varepsilon &= -0.0034, & V_1^\delta &= 0.0023, & V_0^\delta &= -0.0064, & C_2^{\varepsilon, \delta} &= -0.0073, & C_1^{\varepsilon, \delta} &= -0.0171, \\ C_0^{\varepsilon, \delta} &= 0.0183, & C^{\varepsilon, \delta} &= 0.0047, & A_2^\varepsilon &= -0.0002, & A_1^\varepsilon &= 0.0038, & A_0^\varepsilon &= -0.0183, & A^\varepsilon &= 0.0011, \\ B_2^\delta &= 0.0080, & B_1^\delta &= 0.0183, & \frac{V_3'^\varepsilon}{\sigma'} &= 0.0146, & \frac{V_1'^\delta}{\sigma'} &= -0.3104, & \frac{V_0'^\delta}{\sigma'} &= 0.9856, & \phi^\varepsilon &= -0.0181. \end{aligned}$$

In Figure 2 we plot the implied volatility fit from March 18, 2010. The parameters obtained from the above calibration procedure are

$$\begin{aligned} \sigma^* &= 0.2269, & V_3^\varepsilon &= -0.0062, & V_1^\delta &= -0.0026, & V_0^\delta &= 0.0208, & C_2^{\varepsilon, \delta} &= -0.0031, & C_1^{\varepsilon, \delta} &= -0.00034, \\ C_0^{\varepsilon, \delta} &= -0.0035, & C^{\varepsilon, \delta} &= 0.0033, & A_2^\varepsilon &= 0.0034, & A_1^\varepsilon &= 0.0034, & A_0^\varepsilon &= -0.0004, & A^\varepsilon &= -0.0012, \\ B_2^\delta &= 0.0012, & B_1^\delta &= -0.0035, & \frac{V_3'^\varepsilon}{\sigma'} &= -0.1590, & \frac{V_1'^\delta}{\sigma'} &= 0.0914, & \frac{V_0'^\delta}{\sigma'} &= -0.0729, & \phi^\varepsilon &= -0.0443. \end{aligned}$$

Notice that, in both cases, the obtained parameters other than  $\sigma^*$  are small, as expected in the regime of validity of our expansion (i.e., small  $\varepsilon$  and small  $\delta$ ).

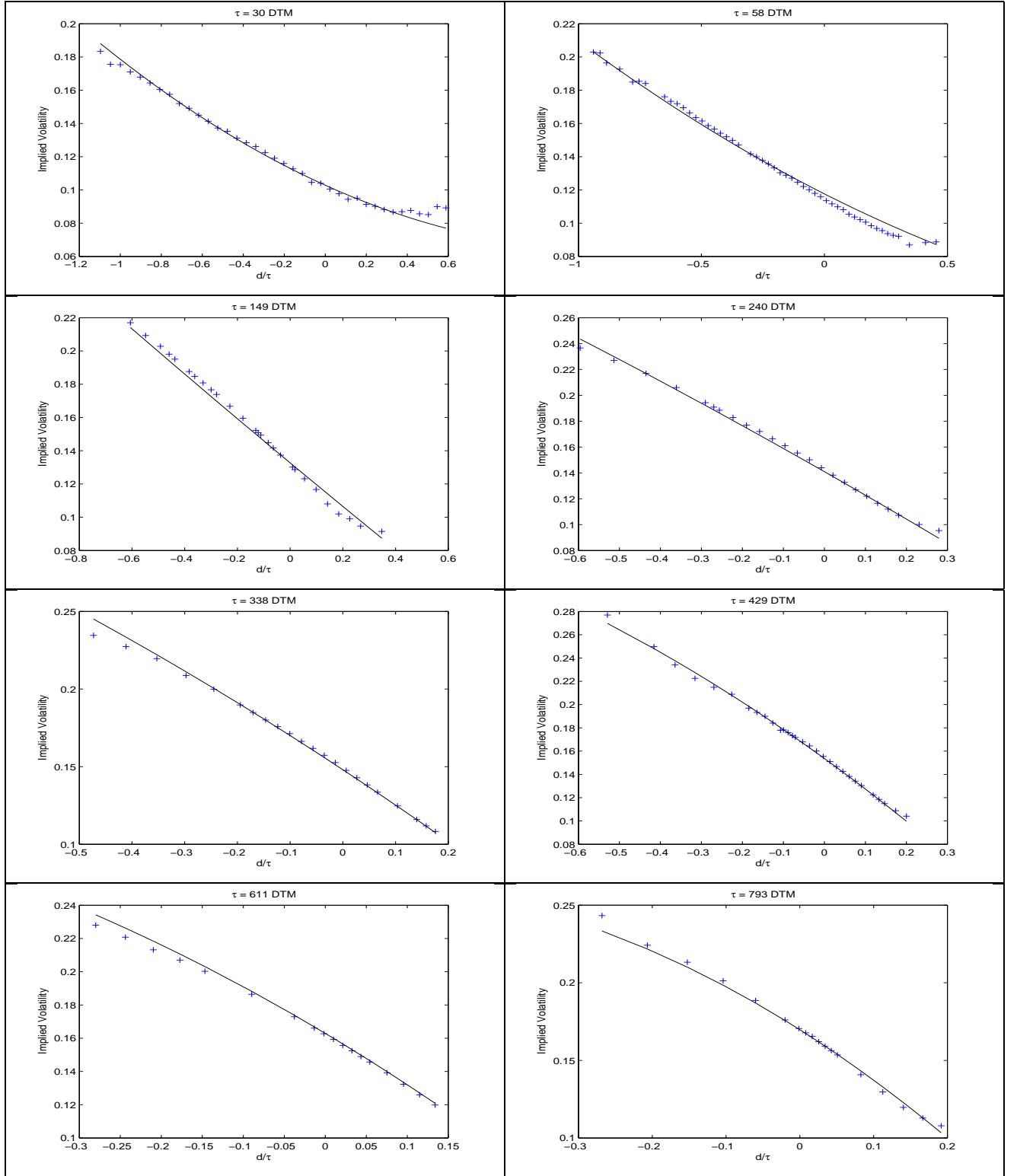


Figure 1: *Implied volatility fit to S&P 500 index options on October 19, 2006. Note that this is the result of a single calibration to all maturities and not a maturity-by-maturity calibration. Each panel shows the DTM=days to maturity.*



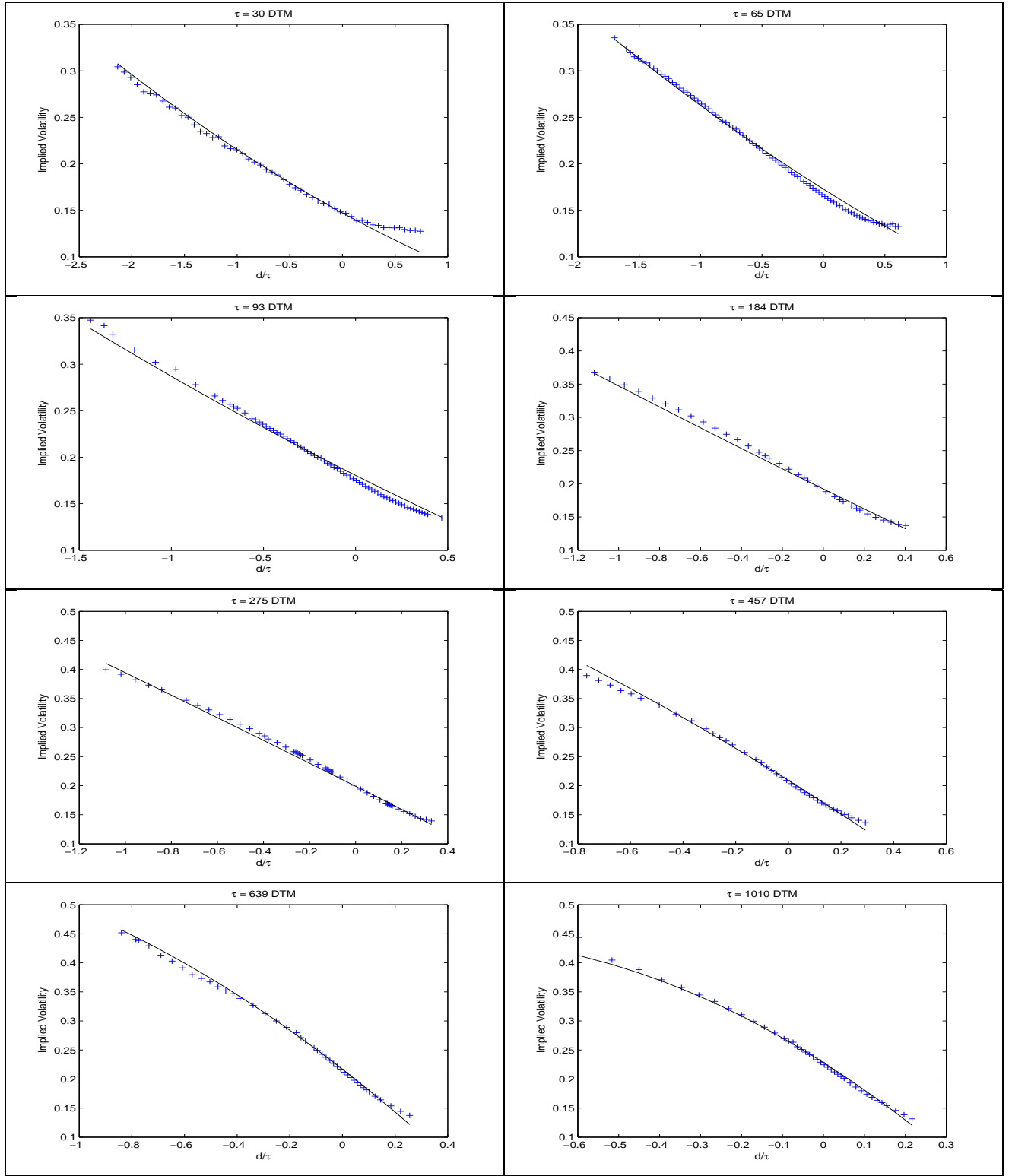


Figure 2: *Implied volatility fit to S&P 500 index options on March 18, 2010. Note that this is the result of a single calibration to all maturities and not a maturity-by-maturity calibration.*

## 4 Concluding Remarks

We have derived a second order asymptotic approximation for European options under multiscale stochastic volatility models with fast and slow factors. Proof of convergence requires a terminal layer analysis that is developed probabilistically, in contrast to the techniques of matched asymptotic expansions that are more common in fluid mechanics. The price approximation is translated to an implied volatility surface approximation which is quadratic in log-moneyness and highly nontrivial in the term structure direction.

In applying asymptotic approximations, two questions are crucial: how is it calibrated to data, and how can the recovered (partial) information be used. Here we have shown that the complicated second order formulas fit the data well across strikes and maturities (Figures 1 and 2). Moreover, the extracted parameters are small when they should be small in the regime of the asymptotic analysis (Section 3.3). These parameters can then be used to value exotic options such as barrier options, that are characterized by the same PDE but with different boundary conditions, to the same order of accuracy, as discussed in Section 2.7.

It remains to investigate *stability* of the calibrated parameters over time, that is, to show empirically all but one of the parameters vary like the slow volatility factor, and the remaining one more rapidly. This detailed data work is in preparation.

## A Proof of Proposition 2.1

The goal is to compute  $\langle \mathcal{L}_2 \rangle F_{2,0}$ . Starting from the expression (2.21) for  $P_{2,0}$ , applying the operator  $\mathcal{L}_2$  and averaging, we obtain:

$$\begin{aligned}
\langle \mathcal{L}_2 P_{2,0} \rangle &= \left\langle \mathcal{L}_2 \left( -\frac{1}{2} \phi \mathcal{D}_2 P_{0,0} + F_{2,0} \right) \right\rangle = -\frac{1}{2} \langle \phi \mathcal{L}_2 \rangle \mathcal{D}_2 P_{0,0} + \langle \mathcal{L}_2 \rangle F_{2,0} \\
&= -\frac{1}{2} \left( \langle \phi \rangle (\partial_t - r) + \frac{1}{2} \langle \phi f^2 \rangle \mathcal{D}_2 \right) \mathcal{D}_2 P_{0,0} + \langle \mathcal{L}_2 \rangle F_{2,0} \\
&= -\frac{1}{2} \left( \langle \phi \rangle \left( \partial_t - r + \frac{1}{2} \langle f^2 \rangle \mathcal{D}_2 \right) + \frac{1}{2} \left( \langle \phi f^2 \rangle - \langle \phi \rangle \langle f^2 \rangle \right) \mathcal{D}_2 \right) \mathcal{D}_2 P_{0,0} + \langle \mathcal{L}_2 \rangle F_{2,0} \\
&= -\frac{1}{2} \langle \phi \rangle \langle \mathcal{L}_2 \rangle \mathcal{D}_2 P_{0,0} - \frac{1}{4} \left( \langle \phi f^2 \rangle - \langle \phi \rangle \langle f^2 \rangle \right) \mathcal{D}_2^2 P_{0,0} + \langle \mathcal{L}_2 \rangle F_{2,0} \\
&= A \mathcal{D}_2^2 P_{0,0} + \langle \mathcal{L}_2 \rangle F_{2,0},
\end{aligned} \tag{A.1}$$

where  $A(z)$  is given in Proposition 2.1.

On the other hand, by (2.16),  $\langle \mathcal{L}_2 P_{2,0} \rangle = -\langle \mathcal{L}_1 P_{3,0} \rangle$ , and therefore we compute  $P_{3,0}$ . From (2.10), (2.15), (2.20), (2.21), and the definitions of  $\mathcal{L}_1$  and  $\mathcal{L}_2$ , we express  $\mathcal{L}_0 P_{3,0}$  as follows

$$\begin{aligned}
\mathcal{L}_0 P_{3,0} &= -(\mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0}) \\
&= -(\mathcal{L}_1 P_{2,0} - \langle \mathcal{L}_1 P_{2,0} \rangle) - (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_{1,0} \\
&= -\mathcal{L}_1 \left( -\frac{1}{2} \phi \mathcal{D}_2 P_{0,0} + F_{2,0} \right) + \left\langle \mathcal{L}_1 \left( -\frac{1}{2} \phi \mathcal{D}_2 P_{0,0} + F_{2,0} \right) \right\rangle - \left( \frac{1}{2} (f^2 - \langle f^2 \rangle) \mathcal{D}_2 P_{1,0} \right) \\
&= - \left( -\frac{1}{2} \rho_{xy} (\beta f \partial_y \phi - \langle \beta f \partial_y \phi \rangle) \mathcal{D}_1 \mathcal{D}_2 + \frac{1}{2} (\beta \Lambda \partial_y \phi - \langle \beta \Lambda \partial_y \phi \rangle) \mathcal{D}_2 \right) P_{0,0} - \left( \frac{1}{2} \mathcal{L}_0 \phi \right) \mathcal{D}_2 P_{1,0}.
\end{aligned}$$

Using the functions  $\psi_1$  and  $\psi_2$  defined in (2.26), the solution to the previous equation is given by

$$P_{3,0} = \frac{1}{2} \rho_{xy} \psi_1 \mathcal{D}_1 \mathcal{D}_2 P_{0,0} - \frac{1}{2} \psi_2 \mathcal{D}_2 P_{0,0} - \frac{1}{2} \phi \mathcal{D}_2 P_{1,0} + F_{3,0},$$

where  $F_{3,0}(t, x, z)$  is independent of  $y$ . Now, we can compute  $\langle \mathcal{L}_1 P_{3,0} \rangle$ :

$$\begin{aligned}
\langle \mathcal{L}_1 P_{3,0} \rangle &= \left\langle \left( \rho_{xy} \beta f \mathcal{D}_1 - \beta \Lambda \right) \partial_y \left( \frac{1}{2} \rho_{xy} \psi_1 \mathcal{D}_1 \mathcal{D}_2 P_{0,0} - \frac{1}{2} \psi_2 \mathcal{D}_2 P_{0,0} - \frac{1}{2} \phi \mathcal{D}_2 P_{1,0} \right) \right\rangle \\
&= \frac{1}{2} \rho_{xy}^2 \langle \beta f \partial_y \psi_1 \rangle \mathcal{D}_1^2 \mathcal{D}_2 P_{0,0} - \frac{1}{2} \rho_{xy} \langle \beta \Lambda \partial_y \psi_1 \rangle \mathcal{D}_1 \mathcal{D}_2 P_{0,0} - \frac{1}{2} \rho_{xy} \langle \beta f \partial_y \psi_2 \rangle \mathcal{D}_1 \mathcal{D}_2 P_{0,0} \\
&\quad + \frac{1}{2} \langle \beta \Lambda \partial_y \psi_2 \rangle \mathcal{D}_2 P_{0,0} - \frac{1}{2} \rho_{xy} \langle \beta f \partial_y \phi \rangle \mathcal{D}_1 \mathcal{D}_2 P_{1,0} + \frac{1}{2} \langle \beta \Lambda \partial_y \phi \rangle \mathcal{D}_2 P_{1,0} \\
&= \left( A_2 \mathcal{D}_1^2 \mathcal{D}_2 + A_1 \mathcal{D}_1 \mathcal{D}_2 + A_0 \mathcal{D}_2 \right) P_{0,0} + \left( V_3 \mathcal{D}_1 \mathcal{D}_2 + V_2 \mathcal{D}_2 \right) P_{1,0},
\end{aligned} \tag{A.2}$$

where  $A_2(z)$ ,  $A_1(z)$  and  $A_0(z)$  are given in Proposition 2.1. Inserting (A.1) and (A.2) into (2.16) yields the following PDE for  $F_{2,0}$

$$\langle \mathcal{L}_2 \rangle F_{2,0} = -\mathcal{A} P_{0,0} - \mathcal{V} P_{1,0},$$

which agrees with (2.25). Lastly, the terminal condition  $F_{2,0}(T, x, z) = 0$  is imposed by averaging (2.21), and using (2.22) and (2.23):

$$\langle P_{2,0}(T, x, \cdot, z) \rangle = -\frac{1}{2} \langle \phi \rangle \mathcal{D}_2 P_{0,0}(T, x, z) + F_{2,0}(T, x, z) = F_{2,0}(T, x, z) = 0.$$

## B Proof of Proposition 2.2

In order to compute  $\langle \mathcal{L}_1 P_{2,1} \rangle$  we first compute  $P_{2,1}$ . Using (2.27), (2.29) and the definition (2.35) of  $\psi_3$  and  $\psi_4$ , we find

$$\begin{aligned}
\mathcal{L}_0 P_{2,1} &= -\mathcal{L}_2 P_{0,1} - \mathcal{M}_1 P_{0,0} \\
&= -(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_{0,1} - (\mathcal{M}_1 - \langle \mathcal{M}_1 \rangle) P_{0,0} \\
&= -\frac{1}{2} (f^2 - \langle f^2 \rangle) \mathcal{D}_2 P_{0,1} - \rho_{xz} g (f - \langle f \rangle) \mathcal{D}_1 \partial_z P_{0,0} + g (\Gamma - \langle \Gamma \rangle) \partial_z P_{0,0} \\
&= -\frac{1}{2} \mathcal{L}_0 \phi \mathcal{D}_2 P_{0,1} - \rho_{xz} g \mathcal{L}_0 \psi_3 \mathcal{D}_1 \partial_z P_{0,0} + g \mathcal{L}_0 \psi_4 \partial_z P_{0,0}.
\end{aligned}$$

Thus,  $P_{2,1}$  is given by

$$P_{2,1} = -\frac{1}{2} \phi \mathcal{D}_2 P_{0,1} - \rho_{xz} g \psi_3 \mathcal{D}_1 \partial_z P_{0,0} + g \psi_4 \partial_z P_{0,0} + F_{2,1}(t, x, z), \tag{B.1}$$

where  $F_{2,1}(t, x, z)$  does not depend on  $y$ . Next, using expression (B.1) for  $P_{2,1}$  we find

$$\begin{aligned}
\langle \mathcal{L}_1 P_{2,1} \rangle &= \left\langle \left( \rho_{xy} \beta f \mathcal{D}_1 - \beta \Lambda \right) \partial_y \left( -\frac{1}{2} \phi \mathcal{D}_2 P_{0,1} \right) \right\rangle \\
&\quad + \left\langle \left( \rho_{xy} \beta f \mathcal{D}_1 - \beta \Lambda \right) \partial_y \left( -\rho_{xz} g \psi_3 \mathcal{D}_1 \partial_z P_{0,0} \right) \right\rangle \\
&\quad + \left\langle \left( \rho_{xy} \beta f \mathcal{D}_1 - \beta \Lambda \right) \partial_y \left( g \psi_4 \partial_z P_{0,0} \right) \right\rangle \\
&= -\frac{1}{2} \rho_{xy} \langle \beta f \partial_y \phi \rangle \mathcal{D}_1 \mathcal{D}_2 P_{0,1} + \frac{1}{2} \langle \beta \Lambda \partial_y \phi \rangle \mathcal{D}_2 P_{0,1} \\
&\quad - \rho_{xy} \rho_{xz} g \langle \beta f \partial_y \psi_3 \rangle \mathcal{D}_1^2 \partial_z P_{0,0} + \rho_{xz} g \langle \beta \Lambda \partial_y \psi_3 \rangle \mathcal{D}_1 \partial_z P_{0,0} \\
&\quad + \rho_{xy} g \langle \beta f \partial_y \psi_4 \rangle \mathcal{D}_1 \partial_z P_{0,0} - g \langle \beta \Lambda \partial_y \psi_4 \rangle \partial_z P_{0,0} \\
&= (V_3 \mathcal{D}_1 \mathcal{D}_2 + V_2 \mathcal{D}_2) P_{0,1} + \frac{1}{\sigma'} (C_2 \mathcal{D}_1^2 + C_1 \mathcal{D}_1 + C_0) \partial_z P_{0,0},
\end{aligned}$$

which agrees with (2.33). Next, using expression (2.21) for  $P_{2,0}$  we find

$$\begin{aligned}\langle \mathcal{M}_3 P_{2,0} \rangle &= \left\langle \left( \rho_{yz} \beta(y) g(z) \partial_{yz}^2 \right) \left( -\frac{1}{2} \phi \mathcal{D}_2 P_{0,0} + F_{2,0} \right) \right\rangle \\ &= -\frac{1}{2} \rho_{yz} g \langle \beta \partial_y \phi \rangle \mathcal{D}_2 \partial_z P_{0,0} \\ &= \frac{1}{\bar{\sigma}'} C \mathcal{D}_2 \partial_z P_{0,0},\end{aligned}$$

which agrees with (2.34).

## C Proof of Proposition 2.3

As previously noted, the operator  $\langle \mathcal{L}_2 \rangle$ , given by (2.17), is the Black-Scholes pricing operator  $\mathcal{L}_{BS}$ , where the Black-Scholes volatility  $\sigma$  has been replaced by an average level of volatility  $\bar{\sigma}$ , and so that  $P_{0,0} = P_{BS}(\bar{\sigma}(z))$ . In order to derive expressions for the higher order  $\{P_{i,j}, 1 \leq i+j \leq 2\}$ , we need the following two lemmas. Recall also the notation  $\tau = T - t$ .

**Lemma C.1.** *For European options,*

$$\partial_\sigma P_{BS}(\sigma) = \tau \sigma \mathcal{D}_2 P_{BS}(\sigma). \quad (\text{C.1})$$

*Proof.* This is the classical relation between Vega and Gamma for plain vanilla European options. It is easily obtained by differentiating the Black-Scholes PDE with respect to  $\sigma$  to obtain a Black-Scholes PDE with source for the Vega which in turn can be solved explicitly in terms of the Gamma.  $\square$

Using Lemma C.1 and the fact that the logarithmic derivative operators  $\mathcal{D}_k$  in (2.3) commute ( $\mathcal{D}_k \mathcal{D}_m = \mathcal{D}_m \mathcal{D}_k$ ), which implies that  $\langle \mathcal{L}_2 \rangle$  and any  $\mathcal{D}_k$  commute ( $\langle \mathcal{L}_2 \rangle \mathcal{D}_k = \mathcal{D}_k \langle \mathcal{L}_2 \rangle$ ), one can show:

**Lemma C.2.** *For European options, and positive integers  $k$  and  $n$ ,*

$$\begin{aligned}\langle \mathcal{L}_2 \rangle \frac{\tau^{n+1}}{n+1} P(\{\mathcal{D}_k\}) P_{BS}(\bar{\sigma}(z)) &= -\tau^n P(\{\mathcal{D}_k\}) P_{BS}(\bar{\sigma}(z)), \\ \langle \mathcal{L}_2 \rangle \frac{\tau^{n+1}}{n+2} P(\{\mathcal{D}_k\}) \partial_\sigma P_{BS}(\bar{\sigma}(z)) &= -\tau^n P(\{\mathcal{D}_k\}) \partial_\sigma P_{BS}(\bar{\sigma}(z)), \\ \langle \mathcal{L}_2 \rangle \frac{\tau^{n+1}}{n+3} P(\{\mathcal{D}_k\}) \left( \partial_{\sigma\sigma}^2 + \frac{1}{\bar{\sigma}(n+2)} \partial_\sigma \right) P_{BS}(\bar{\sigma}(z)) &= -\tau^n P(\{\mathcal{D}_k\}) \partial_{\sigma\sigma}^2 P_{BS}(\bar{\sigma}(z)),\end{aligned}$$

where  $P(\{\mathcal{D}_k\})$  is some polynomial of  $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_k$ .

*Proof.* The proof is a straightforward computation. In showing the second and third relations, the  $\partial_\sigma$  partial derivatives acting on  $P_{BS}$  are first converted into  $\mathcal{D}_2$  using Lemma C.1 which now commute with any  $\mathcal{D}_k$  operators and  $\langle \mathcal{L}_2 \rangle$ . The final step uses that  $\langle \mathcal{L}_2 \rangle P_{BS}(\bar{\sigma}(z)) = 0$ .  $\square$

Using Lemmas C.1 and C.2, a direct computation shows that the  $\{P_{i,j}\}$  of Theorem 2.3 satisfy the PDEs of (2.41) and the appropriate terminal conditions.

## D Proof of the Accuracy Theorem 2.4

In what follows, we will make use several times of the fact that  $P_{0,0}$  and its derivatives  $\mathcal{D}_k P_{0,0}$  are bounded in  $x$  (because of our smoothness and boundedness assumption on the payoff function  $h$  and its derivatives). We will also use the fact that  $Y$  and  $Z$  have moments of all orders uniformly bounded in  $\varepsilon$  and  $\delta$  (thanks

to the assumptions made on  $Y^{(1)}$  and  $Z^{(1)}$  in section 2.6). The proof of this can be found following Lemma 4.1 of [8].

We begin the proof by recalling our price approximation  $\tilde{P}^{\varepsilon, \delta}$  from (2.40):

$$P^{\varepsilon, \delta} \approx \tilde{P}^{\varepsilon, \delta} = P_{0,0} + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1} + \sqrt{\varepsilon \delta} P_{1,1} + \varepsilon P_{2,0} + \delta P_{0,2},$$

where  $\{P_{i,j}, i+j \leq 2\}$  are given in Theorem 2.3. Since the singular perturbation argument involves terms with higher order in  $\varepsilon$ , we introduce

$$\hat{P}^{\varepsilon, \delta} = \tilde{P}^{\varepsilon, \delta} + \varepsilon^{3/2} P_{3,0} + \varepsilon^2 P_{4,0} + \varepsilon \sqrt{\delta} P_{2,1} + \varepsilon^{3/2} \sqrt{\delta} P_{3,1},$$

and we observe that

$$\hat{P}^{\varepsilon, \delta} = \tilde{P}^{\varepsilon, \delta} + \mathcal{O}(\varepsilon^{3/2} + \varepsilon \sqrt{\delta}).$$

Next, we define the residual

$$R^{\varepsilon, \delta} := P^{\varepsilon, \delta} - \hat{P}^{\varepsilon, \delta},$$

and therefore, the proof consists of showing that  $R^{\varepsilon, \delta} = \mathcal{O}(\varepsilon^{3/2} + \varepsilon \sqrt{\delta} + \delta \sqrt{\varepsilon})$ .

From the equations (2.2), (2.9), (2.10), (2.11), (2.27), (2.28), and (2.37) satisfied by  $P^{\varepsilon, \delta}$  and  $\{P_{i,j}, i \leq 4, j \leq 2, i+j \leq 4\}$ , we deduce that the residual  $R^{\varepsilon, \delta}$  satisfies the following PDE:

$$\mathcal{L}^{\varepsilon, \delta} R^{\varepsilon, \delta} + \mathcal{O}(\varepsilon^{3/2} + \varepsilon \sqrt{\delta} + \delta \sqrt{\varepsilon}) = 0. \quad (\text{D.1})$$

From the terminal conditions for  $\{P_{i,j}, i+j \leq 2\}$ , we deduce the terminal condition for the residual:

$$R^{\varepsilon, \delta}(T, x, y, z) = -\varepsilon P_{2,0}(T, x, y, z) + \mathcal{O}(\varepsilon^{3/2} + \varepsilon \sqrt{\delta} + \delta \sqrt{\varepsilon}), \quad (\text{D.2})$$

where it is important to note that the non-vanishing terminal condition  $P_{2,0}(T, x, y, z)$  plays a particular role at the  $\varepsilon$  order. The probabilistic representation of  $R^{\varepsilon, \delta}$ , solution to the Cauchy problem (D.1)-(D.2), is

$$R^{\varepsilon, \delta}(t, x, y, z) = -\varepsilon \tilde{\mathbb{E}}_{t,x,y,z} \left[ e^{-r(T-t)} P_{2,0}(T, X_T, Y_T, Z_T) \right] + \mathcal{O}(\varepsilon^{3/2} + \varepsilon \sqrt{\delta} + \delta \sqrt{\varepsilon}),$$

where  $\tilde{\mathbb{E}}_{t,x,y,z}$  denotes expectation under the  $(\varepsilon, \delta)$ -dependent dynamics (2.1) starting at time  $t < T$  from  $(x, y, z)$ . From the expression  $P_{2,0} = -\frac{1}{2} \phi \mathcal{D}_2 P_{0,0} + F_{2,0}$  in (2.21) and the terminal condition  $F_{2,0}(T, x, z) = 0$  in (2.25), we deduce

$$R^{\varepsilon, \delta}(t, x, y, z) = \frac{\varepsilon}{2} \tilde{\mathbb{E}}_{t,x,y,z} \left[ e^{-r(T-t)} \phi(Y_T, Z_T) \mathcal{D}_2 P_{0,0}(T, X_T, Z_T) \right] + \mathcal{O}(\varepsilon^{3/2} + \varepsilon \sqrt{\delta} + \delta \sqrt{\varepsilon}).$$

As  $\delta \rightarrow 0$ , the process  $Z$  converges to the constant process  $z$ , and as  $\varepsilon \rightarrow 0$ , the process  $X$  converges in distribution to a geometric Brownian motion  $\bar{X}$  with volatility  $\bar{\sigma}(z)$  and independent of the  $Y$  process. Therefore, up to an error term of order  $\mathcal{O}(\varepsilon^{3/2} + \varepsilon \sqrt{\delta} + \delta \sqrt{\varepsilon})$ , we deduce that

$$R^{\varepsilon, \delta}(t, x, y, z) = \varepsilon C \tilde{\mathbb{E}}_{t,y} [\phi(Y_T, z)] + \mathcal{O}(\varepsilon^{3/2} + \varepsilon \sqrt{\delta} + \delta \sqrt{\varepsilon}), \quad (\text{D.3})$$

where the constant

$$C := \tilde{\mathbb{E}}_{t,x} \left[ e^{-r(T-t)} \mathcal{D}_2 P_{0,0}(T, \bar{X}_T, z) \right],$$

does not depend on  $(\varepsilon, \delta)$ .

If  $\Lambda = 0$ , that is, if the process  $Y$  had infinitesimal generator  $\mathcal{L}_0$  under the pricing measure  $\tilde{\mathbb{P}}$ , then due to the ergodic theorem,  $\tilde{\mathbb{E}}_{t,y} [\phi(Y_T, z)]$  would converge to zero exponentially fast as  $\varepsilon \rightarrow 0$  since  $\langle \phi \rangle = 0$ .

This is exactly where we see that our choice of terminal condition for  $P_{2,0}$  was necessary because if  $\langle \phi \rangle \neq 0$ , then the residual would be of order  $\varepsilon$ . However, under the pricing measure  $\tilde{\mathbb{P}}$ , the process  $Y$  does not have generator  $\mathcal{L}_0$  due to the presence of the possibly nonzero market price of volatility risk  $\Lambda(y, z)$  and therefore, we need to analyze the behavior as  $\varepsilon \rightarrow 0$  of  $\tilde{\mathbb{E}}_{t,y}[\phi(Y_T, z)]$  for  $\Lambda \neq 0$ .

Without loss of generality we can assume  $t = 0$ . By rescaling time it is enough to consider  $\tilde{\mathbb{E}}[\phi(Y_{T/\varepsilon}, z)]$  under the autonomous dynamics

$$dY_t = (\alpha(Y_t) - \sqrt{\varepsilon}\beta(Y_t)\Lambda(Y_t, z))dt + \beta(Y_t)d\tilde{W}_t^y, \quad Y_0 = y. \quad (\text{D.4})$$

The process  $Y$  (D.4) admits an invariant distribution  $\Pi_\varepsilon$  with density

$$\pi_\varepsilon(y) = \frac{J_\varepsilon}{\beta^2(y)} \exp\left(2 \int_0^y \frac{\alpha(u) - \sqrt{\varepsilon}\beta(u)\Lambda(u, z)}{\beta^2(u)} du\right),$$

where  $J_\varepsilon$  is a normalization factor.

From the ergodic property of  $Y$  and the assumption of a positive spectral gap, for  $\varepsilon$  small enough, one can find positive constants  $C_1$  and  $\lambda$  such that

$$|\tilde{\mathbb{E}}[\phi(Y_{T/\varepsilon}, z)] - \langle \phi \rangle_\varepsilon| \leq C_1 e^{-\lambda T/\varepsilon},$$

where  $\langle \cdot \rangle_\varepsilon$  denotes averaging with respect to  $\Pi_\varepsilon$ . Now, expanding  $\pi_\varepsilon$  (including  $J_\varepsilon$ ), we derive for any  $g \in L_1(\Pi_\varepsilon)$

$$\langle g \rangle_\varepsilon = \langle g \rangle - 2\sqrt{\varepsilon} \left\langle \left( \int_0^y \frac{\Lambda(u, z)}{\beta(u)} du \right) (g(\cdot) - \langle g \rangle) \right\rangle + \dots \quad (\text{D.5})$$

Hence, using the fact that  $\langle \phi \rangle = 0$  we obtain that there is a constant  $C_2$  such that

$$|\tilde{\mathbb{E}}[\phi(Y_{T/\varepsilon}, z)]| \leq C_2 \sqrt{\varepsilon}.$$

We then conclude from (D.3) that  $R^{\varepsilon, \delta}$  is of  $\mathcal{O}(\varepsilon^{3/2} + \varepsilon\sqrt{\delta} + \delta\sqrt{\varepsilon})$ .

## E Proof of Proposition 2.5

Define  $P_{i,j}^*$  as the solutions to

$$\left. \begin{aligned} \mathcal{O}(1): \quad & \langle \mathcal{L}_2^* \rangle P_{0,0}^* = 0, & P_{0,0}^*(T, x, z) &= h(x), \\ \mathcal{O}(\sqrt{\varepsilon}): \quad & \langle \mathcal{L}_2^* \rangle P_{1,0}^* = -\mathcal{V}^* P_{0,0}^*, & P_{1,0}^*(T, x, z) &= 0, \\ \mathcal{O}(\sqrt{\delta}): \quad & \langle \mathcal{L}_2^* \rangle P_{0,1}^* = -\langle \mathcal{M}_1 \rangle P_{0,0}^*, & P_{0,1}^*(T, x, z) &= 0, \\ \mathcal{O}(\varepsilon): \quad & P_{2,0}^* = -\frac{1}{2} \phi \mathcal{D}_2 P_{0,0}^* + F_{2,0}^*, & & \\ & \langle \mathcal{L}_2^* \rangle F_{2,0}^* = -\mathcal{A} P_{0,0}^* - \mathcal{V}^* P_{1,0}^*, & F_{2,0}^*(T, x, z) &= 0, \\ \mathcal{O}(\delta): \quad & \langle \mathcal{L}_2^* \rangle P_{0,2}^* = -\langle \mathcal{M}_1 \rangle P_{0,1}^* - \mathcal{M}_2 P_{0,0}^*, & P_{0,2}^*(T, x, z) &= 0, \\ \mathcal{O}(\sqrt{\varepsilon\delta}): \quad & \langle \mathcal{L}_2^* \rangle P_{1,1}^* = -\mathcal{V}^* P_{0,1}^* - \frac{1}{\sigma'} \mathcal{C} \partial_z P_{0,0}^* - \langle \mathcal{M}_1 \rangle P_{1,0}^*, & P_{1,1}^*(T, x, z) &= 0, \end{aligned} \right\}$$

where

$$\langle \mathcal{L}_2^* \rangle := \langle \mathcal{L}_2 \rangle + \sqrt{\varepsilon} V_2 \mathcal{D}_2, \quad \mathcal{V}^* := \mathcal{V} - V_2 \mathcal{D}_2.$$

These correspond to terms in (2.41) of the asymptotic approximation to second order where  $\bar{\sigma}(z)$  is replaced by  $\sigma^*(z)$  defined in (2.46), and the terms containing  $V_2$  are removed. We show that these changes alter the accuracy of the approximation only at higher order.

First, we note that  $(P_{0,0} - P_{0,0}^*) = \mathcal{O}(\sqrt{\varepsilon})$  since

$$\langle \mathcal{L}_2 \rangle (P_{0,0} - P_{0,0}^*) = \sqrt{\varepsilon} V_2 \mathcal{D}_2 P_{0,0}^*, \quad P_{0,0}(T, x, z) - P_{0,0}^*(T, x, z) = 0.$$

Next, we define  $E_1^{\varepsilon, \delta}(t, x, z)$  by

$$E_1^{\varepsilon, \delta} := \left( P_{0,0} + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1} \right) - \left( P_{0,0}^* + \sqrt{\varepsilon} P_{1,0}^* + \sqrt{\delta} P_{0,1}^* \right),$$

the difference in the first order approximations. Note that  $E_1^{\varepsilon, \delta}(T, x, z) = 0$  and

$$\langle \mathcal{L}_2 \rangle E_1^{\varepsilon, \delta} = \left[ \sqrt{\varepsilon} (\mathcal{V}^* + V_2 \mathcal{D}_2) + \sqrt{\delta} \langle \mathcal{M}_1 \rangle \right] (P_{0,0}^* - P_{0,0}) + \varepsilon V_2 \mathcal{D}_2 P_{1,0}^* + \sqrt{\varepsilon} \delta V_2 \mathcal{D}_2 P_{0,1}^*.$$

Thus, we conclude that  $E_1^{\varepsilon, \delta} = \mathcal{O}(\varepsilon + \sqrt{\varepsilon} \delta)$ .

Similarly incorporating the order  $\varepsilon$  term, we define  $E_2^\varepsilon(t, x, y, z)$  by

$$E_2^\varepsilon := (P_{0,0} + \sqrt{\varepsilon} P_{1,0} + \varepsilon P_{2,0}) - (P_{0,0}^* + \sqrt{\varepsilon} P_{1,0}^* + \varepsilon P_{2,0}^*).$$

From equation (D.5) and by using  $\mathcal{D}_2 (P_{0,0} - P_{0,0}^*) = \mathcal{O}(\sqrt{\varepsilon})$  one can show that  $E_2^\varepsilon(T, x, y, z) = \mathcal{O}(\varepsilon^{3/2})$ . We then compute

$$\langle \mathcal{L}_2 \rangle E_2^\varepsilon = \sqrt{\varepsilon} \mathcal{V} [(P_{0,0}^* + \sqrt{\varepsilon} P_{1,0}^*) - (P_{0,0} + \sqrt{\varepsilon} P_{1,0})] + \varepsilon \mathcal{A} (P_{0,0}^* - P_{0,0}) + \varepsilon^{3/2} V_2 \mathcal{D}_2 P_{2,0}^*.$$

Incorporating the order  $\sqrt{\varepsilon} \delta$  term, we define  $E_3^\varepsilon(t, x, z)$  by

$$E_3^\varepsilon := (P_{0,1} + \sqrt{\varepsilon} P_{1,1}) - (P_{0,1}^* + \sqrt{\varepsilon} P_{1,1}^*).$$

Note that  $E_3^\varepsilon(T, x, z) = 0$  and

$$\langle \mathcal{L}_2 \rangle E_3^\varepsilon = \langle \mathcal{M}_1 \rangle [(P_{0,0}^* + \sqrt{\varepsilon} P_{1,0}^*) - (P_{0,0} + \sqrt{\varepsilon} P_{1,0})] + \sqrt{\varepsilon} \frac{1}{\bar{\sigma}'} \mathcal{C} \partial_z (P_{0,0}^* - P_{0,0}) + \sqrt{\varepsilon} \mathcal{V} (P_{0,1}^* - P_{0,1}).$$

Now define  $E_4^\varepsilon(t, x, z)$  by

$$E_4^\varepsilon := P_{0,2} - P_{0,2}^*.$$

Note that  $E_4^\varepsilon(T, x, z) = 0$  and

$$\langle \mathcal{L}_2 \rangle E_4^\varepsilon = \langle \mathcal{M}_1 \rangle (P_{0,1}^* - P_{0,1}) + \mathcal{M}_2 (P_{0,0}^* - P_{0,0}) + \sqrt{\varepsilon} V_2 \mathcal{D}_2 P_{0,2}^*.$$

Finally,

$$\begin{aligned} \langle \mathcal{L}_2 \rangle (E_2^\varepsilon + \sqrt{\delta} E_3^\varepsilon + \delta E_4^\varepsilon) &= \left( \sqrt{\varepsilon} \mathcal{V} + \sqrt{\delta} \langle \mathcal{M}_1 \rangle \right) E_1^{\varepsilon, \delta} + \varepsilon^{3/2} V_2 \mathcal{D}_2 P_{2,0}^* + \sqrt{\varepsilon} \delta V_2 \mathcal{D}_2 P_{0,2}^* \\ &\quad + \left( \varepsilon \mathcal{A} + \sqrt{\varepsilon} \delta \frac{1}{\bar{\sigma}'} \mathcal{C} \partial_z \right) (P_{0,0}^* - P_{0,0}) + \delta \mathcal{M}_2 (P_{0,0}^* - P_{0,0}). \end{aligned}$$

Hence, we conclude

$$E_2^\varepsilon + \sqrt{\delta} E_3^\varepsilon + \delta E_4^\varepsilon = \mathcal{O}(\varepsilon^{3/2} + \varepsilon \sqrt{\delta} + \sqrt{\varepsilon} \delta).$$

Thus, in (3.3) we can make the following replacements

$$\bar{\sigma} \mapsto \sigma^* := \sqrt{\bar{\sigma}^2 + 2V_2^\varepsilon}, \quad V_2 \mapsto 0.$$

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